

Question 1

Assume that f is integrable on the interval $[a, b]$ and that c is a constant. Prove that

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx.$$

Question 1 - Solution

We prove this property using the definition of the definite integral in terms of Riemann sums.

Let P be a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f on $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

A Riemann sum for cf on $[a, b]$ is

$$\sum_{i=1}^n cf(x_i^*) \Delta x_i.$$

Factor out the constant c :

$$\sum_{i=1}^n cf(x_i^*) \Delta x_i = c \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$. Since f is integrable on $[a, b]$, the limit of its Riemann sums exists:

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx.$$

Therefore,

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n cf(x_i^*) \Delta x_i = c \int_a^b f(x) dx.$$

By definition of the definite integral, this limit equals

$$\int_a^b cf(x) dx.$$

Hence,

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

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Question 2

Assume that f and g are integrable on the interval $[a, b]$. Prove that

$$\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.$$

Question 2 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$, and let $\Delta x_i = x_i - x_{i-1}$. Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for $f + g$ is

$$\sum_{i=1}^n (f(x_i^*) + g(x_i^*)) \Delta x_i.$$

Using algebra, split the sum:

$$\sum_{i=1}^n (f(x_i^*) + g(x_i^*)) \Delta x_i = \sum_{i=1}^n f(x_i^*) \Delta x_i + \sum_{i=1}^n g(x_i^*) \Delta x_i.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$.

Since f and g are integrable on $[a, b]$, the limits of both sums exist:

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx,$$

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n g(x_i^*) \Delta x_i = \int_a^b g(x) dx.$$

Therefore,

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

$$\boxed{\int_a^b (f + g) = \int_a^b f + \int_a^b g}$$

Question 3

Assume that f is integrable on $[a, b]$ and that $a < b$. Prove that

$$\int_a^a f(x) dx = 0.$$

Question 3 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Consider the interval $[a, a]$. Any partition of this interval consists of a single point:

$$a = x_0 = x_1.$$

The width of the subinterval is

$$\Delta x = x_1 - x_0 = a - a = 0.$$

Choose any sample point x^* in the interval $[a, a]$.

A Riemann sum for f over $[a, a]$ is

$$\sum f(x^*) \Delta x = f(x^*) \cdot 0 = 0.$$

Since every Riemann sum over $[a, a]$ equals 0, the limit of the Riemann sums exists and equals 0.

By definition of the definite integral,

$$\int_a^a f(x) dx = 0.$$

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Question 4

Assume that f is integrable on $[a, b]$. Prove that

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.$$

Question 4 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$, with subinterval widths

$$\Delta x_i = x_i - x_{i-1}.$$

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Now consider the interval $[b, a]$. Reverse the order of the partition:

$$b = x_n > x_{n-1} > \cdots > x_0 = a.$$

The corresponding subinterval widths are

$$\Delta x'_i = x_{i-1} - x_i = -\Delta x_i.$$

A Riemann sum for f over $[b, a]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x'_i = \sum_{i=1}^n f(x_i^*) (-\Delta x_i) = - \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Take the limit as the norm of the partition goes to zero. Since f is integrable on $[a, b]$, the limit exists:

$$\int_b^a f(x) dx = - \int_a^b f(x) dx.$$

Rewriting,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

$$\boxed{\int_a^b f(x) dx = - \int_b^a f(x) dx}$$

Question 5

Assume that f is integrable on $[a, b]$ and that

$$f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \geq 0.$$

Question 5 - Solution

We use the definition of the definite integral in terms of Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$, and let

$$\Delta x_i = x_i - x_{i-1}.$$

Choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f on $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Since $f(x) \geq 0$ for all $x \in [a, b]$ and $\Delta x_i > 0$, each term in the sum satisfies

$$f(x_i^*) \Delta x_i \geq 0.$$

Thus, every Riemann sum for f over $[a, b]$ is nonnegative:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i \geq 0.$$

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$. Since f is integrable, the limit of the Riemann sums exists and equals the definite integral:

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Because every sum in the limit is nonnegative, the limit itself must be nonnegative.

Therefore,

$$\int_a^b f(x) dx \geq 0.$$

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Question 6

Assume that f and g are integrable on $[a, b]$ and that

$$f(x) \leq g(x) \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$$

Question 6 - Solution

Define a new function

$$h(x) = g(x) - f(x).$$

Since f and g are integrable on $[a, b]$, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq g(x)$ for all $x \in [a, b]$, we have

$$h(x) = g(x) - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (g(x) - f(x)) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b g(x) \, dx - \int_a^b f(x) \, dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.$$

$$\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$$

Question 7

Assume that f is integrable on $[a, b]$. Prove that

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

Question 7 - Solution

We begin with a basic inequality involving absolute values.

For all real numbers u , we have

$$-|u| \leq u \leq |u|.$$

Apply this inequality to $f(x)$:

$$-|f(x)| \leq f(x) \leq |f(x)| \quad \text{for all } x \in [a, b].$$

Integrate each part of the inequality over $[a, b]$:

$$\int_a^b -|f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

Simplify the left-hand integral:

$$-\int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

This shows that the value of $\int_a^b f(x) \, dx$ lies between $-\int_a^b |f(x)| \, dx$ and $\int_a^b |f(x)| \, dx$.

By definition of absolute value, this implies

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

$$\boxed{\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx}$$

Question 8

Assume that f is integrable on $[a, b]$ and that c satisfies

$$a < c < b.$$

Prove that

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

Question 8 - Solution

We prove this property using Riemann sums.

Let

$$a = x_0 < x_1 < \cdots < x_k = c < \cdots < x_n = b$$

be a partition of $[a, b]$ that includes the point c .

Let

$$\Delta x_i = x_i - x_{i-1}$$

and choose a sample point x_i^* in each subinterval $[x_{i-1}, x_i]$.

A Riemann sum for f over $[a, b]$ is

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

Split this sum at c :

$$\sum_{i=1}^n f(x_i^*) \Delta x_i = \sum_{i=1}^k f(x_i^*) \Delta x_i + \sum_{i=k+1}^n f(x_i^*) \Delta x_i.$$

The first sum is a Riemann sum for f over $[a, c]$, and the second sum is a Riemann sum for f over $[c, b]$.

Now take the limit as the norm of the partition $\|P\| \rightarrow 0$.

Since f is integrable on $[a, b]$, it is integrable on both subintervals $[a, c]$ and $[c, b]$.

Thus,

$$\begin{aligned} \lim_{\|P\| \rightarrow 0} \sum_{i=1}^k f(x_i^*) \Delta x_i &= \int_a^c f(x) dx, \\ \lim_{\|P\| \rightarrow 0} \sum_{i=k+1}^n f(x_i^*) \Delta x_i &= \int_c^b f(x) dx. \end{aligned}$$

Therefore,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Question 9

Assume that f is integrable on $[a, b]$ and that there exists a real number m such that

$$f(x) \geq m \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \geq m(b - a).$$

Question 9 - Solution

Define a new function

$$h(x) = f(x) - m.$$

Since f is integrable on $[a, b]$ and m is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \geq m$ for all $x \in [a, b]$, we have

$$h(x) = f(x) - m \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (f(x) - m) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b f(x) \, dx - \int_a^b m \, dx \geq 0.$$

Since $\int_a^b m \, dx = m(b - a)$, we obtain

$$\int_a^b f(x) \, dx - m(b - a) \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \geq m(b - a).$$

$$\int_a^b f(x) \, dx \geq m(b - a)$$

Question 10

Assume that f is integrable on $[a, b]$ and that there exists a real number M such that

$$f(x) \leq M \quad \text{for all } x \in [a, b].$$

Prove that

$$\int_a^b f(x) \, dx \leq M(b - a).$$

Question 10 - Solution

Define the function

$$h(x) = M - f(x).$$

Since f is integrable on $[a, b]$ and M is a constant, the function h is also integrable on $[a, b]$.

From the assumption $f(x) \leq M$ for all $x \in [a, b]$, we have

$$h(x) = M - f(x) \geq 0 \quad \text{for all } x \in [a, b].$$

By the positivity property of integrals,

$$\int_a^b h(x) \, dx \geq 0.$$

Substitute back for $h(x)$:

$$\int_a^b (M - f(x)) \, dx \geq 0.$$

Use linearity of the integral:

$$\int_a^b M \, dx - \int_a^b f(x) \, dx \geq 0.$$

Since

$$\int_a^b M \, dx = M(b - a),$$

we obtain

$$M(b - a) - \int_a^b f(x) \, dx \geq 0.$$

Rearranging gives

$$\int_a^b f(x) \, dx \leq M(b - a).$$

$$\int_a^b f(x) \, dx \leq M(b - a)$$