

Question 1

Assume that f has a local maximum at $x = a$ and that f is differentiable at $x = a$. Prove that

$$f'(a) = 0.$$

Question 1 - Solution

Since f has a local maximum at $x = a$, there exists $\delta > 0$ such that

$$f(x) \leq f(a) \quad \text{for all } x \text{ with } |x - a| < \delta.$$

Consider the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

for small values of h .

Case 1: $h > 0$

If $0 < h < \delta$, then $a + h$ lies within the interval where $f(a)$ is a local maximum. Hence,

$$f(a+h) - f(a) \leq 0.$$

Since $h > 0$, dividing by h preserves the inequality:

$$\frac{f(a+h) - f(a)}{h} \leq 0.$$

Case 2: $h < 0$

If $-\delta < h < 0$, then again $a + h$ lies within the interval of the local maximum, so

$$f(a+h) - f(a) \leq 0.$$

Since $h < 0$, dividing by h reverses the inequality:

$$\frac{f(a+h) - f(a)}{h} \geq 0.$$

Step 3: Take the limit

For $h > 0$, the difference quotient is less than or equal to 0. For $h < 0$, the difference quotient is greater than or equal to 0.

Since f is differentiable at a , the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists and must be the same from both sides.

The only number that is both ≤ 0 and ≥ 0 is 0.

Therefore,

$$f'(a) = 0.$$

$$\boxed{f'(a) = 0}$$

Question 2

Assume that f is continuous on $[a, b]$, differentiable on (a, b) , and

$$f(a) = f(b).$$

Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = 0.$$

Question 2 - Solution

Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem guarantees that f attains both a maximum and a minimum on $[a, b]$.

Let

$$M = \max_{x \in [a, b]} f(x), \quad m = \min_{x \in [a, b]} f(x).$$

If $M = m$, then f is constant on $[a, b]$, and hence

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

In particular, there exists $c \in (a, b)$ such that $f'(c) = 0$.

Now assume that $M \neq m$. Since $f(a) = f(b)$, the maximum or minimum cannot occur at both endpoints.

Thus, at least one of the extreme values M or m must occur at some point $c \in (a, b)$.

At such a point c , the function f has either a local maximum or a local minimum. Because f is differentiable at c , the derivative must vanish there.

Therefore,

$$f'(c) = 0.$$

There exists $c \in (a, b)$ such that $f'(c) = 0$

Question 3

Assume that f is continuous on $[a, b]$ and differentiable on (a, b) . Prove that there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Question 3 - Solution

Define a new function

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} \right) (x - a).$$

The function g is continuous on $[a, b]$ and differentiable on (a, b) because it is formed from functions with these properties.

Evaluate g at the endpoints:

$$g(a) = f(a) - \left(\frac{f(b) - f(a)}{b - a} \right) (a - a) = f(a),$$

$$g(b) = f(b) - \left(\frac{f(b) - f(a)}{b - a} \right) (b - a) = f(b) - (f(b) - f(a)) = f(a).$$

Thus,

$$g(a) = g(b).$$

By Rolle's Theorem, there exists a number $c \in (a, b)$ such that

$$g'(c) = 0.$$

Differentiate $g(x)$:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Substitute $x = c$:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Solve for $f'(c)$:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Question 4

Assume that f is differentiable on an interval (a, b) and that

$$f'(x) > 0 \quad \text{for all } x \in (a, b).$$

Prove that f is increasing on (a, b) .

Question 4 - Solution

To show that f is increasing on (a, b) , we must prove that for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have

$$f(x_1) < f(x_2).$$

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is differentiable on (a, b) , it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) .

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) > 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Multiply both sides by $x_2 - x_1$:

$$f(x_2) - f(x_1) > 0.$$

Thus,

$$f(x_2) > f(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the function f is increasing on (a, b) .

f is increasing on (a, b)

Question 5

Assume that f is differentiable on an interval (a, b) and that

$$f'(x) < 0 \quad \text{for all } x \in (a, b).$$

Prove that f is decreasing on (a, b) .

Question 5 - Solution

To prove that f is decreasing on (a, b) , we must show that for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, the inequality

$$f(x_1) > f(x_2)$$

holds.

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Because f is differentiable on (a, b) , it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) .

By the Mean Value Theorem, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) < 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0.$$

Multiply both sides by $x_2 - x_1$:

$$f(x_2) - f(x_1) < 0.$$

Thus,

$$f(x_2) < f(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the function f is decreasing on (a, b) .

f is decreasing on (a, b)

Question 6

Assume that f is twice differentiable on an interval (a, b) and that

$$f''(x) > 0 \quad \text{for all } x \in (a, b).$$

Prove that f is concave up on (a, b) .

Question 6 - Solution

To show that f is concave up on (a, b) , we must prove that the derivative $f'(x)$ is increasing on (a, b) .

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is twice differentiable on (a, b) , the function f' is differentiable on (a, b) and therefore continuous on $[x_1, x_2]$.

Apply the Mean Value Theorem to f' on the interval $[x_1, x_2]$. There exists a number $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}.$$

By assumption, $f''(c) > 0$ and since $x_2 - x_1 > 0$, it follows that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} > 0.$$

Multiply both sides by $x_2 - x_1$:

$$f'(x_2) - f'(x_1) > 0.$$

Thus,

$$f'(x_2) > f'(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the derivative $f'(x)$ is increasing on (a, b) .

Therefore, the function f is concave up on (a, b) .

$f \text{ is concave up on } (a, b)$

Question 7

Assume that f is twice differentiable on an interval (a, b) and that

$$f''(x) < 0 \quad \text{for all } x \in (a, b).$$

Prove that f is concave down on (a, b) .

Question 7 - Solution

To show that f is concave down on (a, b) , we must show that the derivative $f'(x)$ is decreasing on (a, b) .

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is twice differentiable on (a, b) , the function f' is differentiable on (a, b) and therefore continuous on $[x_1, x_2]$.

Apply the Mean Value Theorem to f' on the interval $[x_1, x_2]$. There exists a number $c \in (x_1, x_2)$ such that

$$f''(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1}.$$

By assumption, $f''(c) < 0$, and since $x_2 - x_1 > 0$, it follows that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} < 0.$$

Multiply both sides by $x_2 - x_1$:

$$f'(x_2) - f'(x_1) < 0.$$

Thus,

$$f'(x_2) < f'(x_1).$$

Since this holds for all $x_1 < x_2$ in (a, b) , the derivative $f'(x)$ is decreasing on (a, b) .

Therefore, the function f is concave down on (a, b) .

$f \text{ is concave down on } (a, b)$

Question 8

Assume that f is differentiable on an open interval I containing a , and that

$$f'(a) = 0 \quad \text{and} \quad f''(a) > 0.$$

Prove that f has a strict local minimum at $x = a$.

Question 8 - Solution

By the definition of the second derivative and $f'(a) = 0$,

$$\lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a)}{h} = f''(a).$$

Because this limit is nonzero, the quotient has the same sign as $f''(a)$ for all sufficiently small nonzero h .

Consequently, $f'(a+h) < 0$ for negative h and $f'(a+h) > 0$ for positive h . Thus f is strictly decreasing just left of a and strictly increasing just right of a , by the Mean Value Theorem.

Therefore f has a strict local minimum at a . No continuity assumption on f'' near a is needed.

Question 9

Assume that f is continuous on $[a, b]$, differentiable on (a, b) , and that

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

Prove that f is constant on $[a, b]$.

Question 9 - Solution

Let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. We will show that

$$f(x_1) = f(x_2).$$

Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) , the Mean Value Theorem applies. Thus, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption, $f'(c) = 0$, so

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $x_2 - x_1 \neq 0$, it follows that

$$f(x_2) - f(x_1) = 0.$$

Therefore,

$$f(x_2) = f(x_1).$$

Because x_1 and x_2 were arbitrary points in $[a, b]$, we conclude that $f(x)$ has the same value at every point in the interval.

f is constant on $[a, b]$

Question 10

Assume that f is differentiable on an open interval I containing a , and that

$$f'(a) = 0 \quad \text{and} \quad f''(a) < 0.$$

Prove that f has a strict local maximum at $x = a$.

Question 10 - Solution

By the definition of the second derivative and $f'(a) = 0$,

$$\lim_{h \rightarrow 0} \frac{f'(a+h)}{h} = f''(a).$$

Because this limit is nonzero, the quotient has the same sign as $f''(a)$ for all sufficiently small nonzero h .

Consequently, $f'(a+h) > 0$ for negative h and $f'(a+h) < 0$ for positive h . Thus f is strictly increasing just left of a and strictly decreasing just right of a , by the Mean Value Theorem.

Therefore f has a strict local maximum at a . No continuity assumption on f'' near a is needed.