

Question 1

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Question 1 - Solution

We prove this limit using geometric inequalities.

Consider a unit circle centered at the origin.

Let $0 < x < \pi/2$ be measured in radians, and consider the angle x .

The area of the sector with angle x is

$$\text{Area of sector} = \frac{1}{2}x.$$

The area of the triangle formed by the radius and the angle x is

$$\text{Area of triangle} = \frac{1}{2} \sin x.$$

The area of the triangle formed by the tangent line is

$$\text{Area of outer triangle} = \frac{1}{2} \tan x.$$

Thus, the areas satisfy

$$\frac{1}{2} \sin x < \frac{1}{2}x < \frac{1}{2} \tan x.$$

Multiply through by 2:

$$\sin x < x < \tan x.$$

Divide all parts by $\sin x$ (which is positive for $0 < x < \pi/2$):

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Taking reciprocals reverses the inequalities:

$$\cos x < \frac{\sin x}{x} < 1.$$

Now take the limit as $x \rightarrow 0^+$.

Since

$$\lim_{x \rightarrow 0} \cos x = 1,$$

the Squeeze Theorem gives

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1.$$

A similar argument applies for $x \rightarrow 0^-$, so the two-sided limit exists.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Question 2

Prove that

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

Question 2 - Solution

We begin by rewriting the expression using a trigonometric identity.

Recall the identity

$$1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right).$$

Substitute this into the limit:

$$\frac{1 - \cos x}{x} = \frac{2 \sin^2\left(\frac{x}{2}\right)}{x}.$$

Rewrite the fraction:

$$\frac{2 \sin^2\left(\frac{x}{2}\right)}{x} = \sin\left(\frac{x}{2}\right) \cdot \frac{2 \sin\left(\frac{x}{2}\right)}{x}.$$

Now rewrite the second factor:

$$\frac{2 \sin\left(\frac{x}{2}\right)}{x} = \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Thus,

$$\frac{1 - \cos x}{x} = \sin\left(\frac{x}{2}\right) \cdot \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.$$

Now take limits of each factor as $x \rightarrow 0$.

Since $\lim_{u \rightarrow 0} \sin u = 0$, we have

$$\lim_{x \rightarrow 0} \sin\left(\frac{x}{2}\right) = 0.$$

Also, using the fundamental trig limit

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1,$$

we obtain

$$\lim_{x \rightarrow 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = 1.$$

Therefore, by the product law for limits,

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \cdot 1 = 0.$$

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Question 3

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3.$$

Question 3 - Solution

We begin by rewriting the expression so that the fundamental trigonometric limit appears.

Write

$$\frac{\sin(3x)}{x} = \frac{\sin(3x)}{3x} \cdot 3.$$

Now take limits as $x \rightarrow 0$.

Since $3x \rightarrow 0$ as $x \rightarrow 0$, the fundamental trigonometric limit gives

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \left(\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \right) \cdot 3 = 1 \cdot 3 = 3.$$

□

Question 4

Prove that

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

Question 4 - Solution

We rewrite the expression in terms of sine and cosine.

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute this into the limit:

$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \left(\frac{\sin x}{x} \right) \left(\frac{1}{\cos x} \right).$$

Now take limits of each factor as $x \rightarrow 0$.

From the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Also, since cosine is continuous at 0,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1,$$

and therefore

$$\lim_{x \rightarrow 0} \frac{1}{\cos x} = 1.$$

By the product law for limits,

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \cdot 1 = 1.$$

□

Question 5

Prove that

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x^2} = 2.$$

Question 5 - Solution

We begin by using a trigonometric identity.

Recall that

$$1 - \cos(2x) = 2 \sin^2 x.$$

Substitute this into the expression:

$$\frac{1 - \cos(2x)}{x^2} = \frac{2 \sin^2 x}{x^2}.$$

Rewrite the fraction:

$$\frac{2 \sin^2 x}{x^2} = 2 \left(\frac{\sin x}{x} \right)^2.$$

Now take limits as $x \rightarrow 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 = 1^2 = 1.$$

Multiply by 2:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x^2} = 2 \cdot 1 = 2.$$

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Question 6

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x} = 0.$$

Question 6 - Solution

We begin by factoring the expression.

Rewrite the fraction as

$$\frac{\sin x - x}{x} = \frac{\sin x}{x} - 1.$$

Now take the limit as $x \rightarrow 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} - 1 \right) = 1 - 1 = 0.$$

Hence,

$$\boxed{\lim_{x \rightarrow 0} \frac{\sin x - x}{x} = 0}.$$

Question 7

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin x}{\tan x} = 1.$$

Question 7 - Solution

Rewrite the expression using the identity

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute into the fraction:

$$\frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}} = \cos x,$$

for all x where $\tan x$ is defined and $\sin x \neq 0$.

Now take the limit as $x \rightarrow 0$.

Since cosine is continuous at 0,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin x}{\tan x} = 1.$$

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Question 8

Prove that

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0.$$

Question 8 - Solution

We begin by rewriting the expression using a trigonometric identity.

Recall that

$$\cos x - 1 = -(1 - \cos x).$$

Thus,

$$\frac{\cos x - 1}{x} = -\frac{1 - \cos x}{x}.$$

From a previously established trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = -\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = -0 = 0.$$

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Question 9

Prove that

$$\lim_{x \rightarrow 0} \frac{\sin(2x)}{\tan x} = 2.$$

Question 9 - Solution

We begin by rewriting the expression using trigonometric identities.

Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

Substitute this into the fraction:

$$\frac{\sin(2x)}{\tan x} = \sin(2x) \cdot \frac{\cos x}{\sin x}.$$

Next, use the identity

$$\sin(2x) = 2 \sin x \cos x.$$

Substitute:

$$\frac{\sin(2x)}{\tan x} = (2 \sin x \cos x) \cdot \frac{\cos x}{\sin x}.$$

Cancel $\sin x$ (for $x \neq 0$):

$$\frac{\sin(2x)}{\tan x} = 2 \cos^2 x.$$

Now take the limit as $x \rightarrow 0$.

Since cosine is continuous,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1.$$

Therefore,

$$\lim_{x \rightarrow 0} 2 \cos^2 x = 2(1)^2 = 2.$$

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Question 10

Prove that

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x} = 0.$$

Question 10 - Solution

We begin by rewriting the expression to isolate a known limit.

Factor out x from the numerator:

$$\frac{\tan x - x}{x} = \frac{\tan x}{x} - 1.$$

Now rewrite $\tan x$ in terms of sine and cosine:

$$\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \left(\frac{\sin x}{x} \right) \left(\frac{1}{\cos x} \right).$$

Take limits of each factor as $x \rightarrow 0$.

Using the fundamental trigonometric limit,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Since cosine is continuous at 0,

$$\lim_{x \rightarrow 0} \cos x = \cos 0 = 1, \quad \text{so} \quad \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1.$$

Thus,

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

Finally,

$$\lim_{x \rightarrow 0} \left(\frac{\tan x}{x} - 1 \right) = 1 - 1 = 0.$$

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