

Question 1

Assume that f and g are differentiable at $x = a$. Prove that

$$(f + g)'(a) = f'(a) + g'(a).$$

Question 1 - Solution

By definition of the derivative,

$$(f + g)'(a) = \lim_{h \rightarrow 0} \frac{(f + g)(a + h) - (f + g)(a)}{h}.$$

Expand the numerator:

$$(f + g)(a + h) - (f + g)(a) = (f(a + h) + g(a + h)) - (f(a) + g(a)).$$

Group terms:

$$= (f(a + h) - f(a)) + (g(a + h) - g(a)).$$

Substitute back into the limit:

$$(f + g)'(a) = \lim_{h \rightarrow 0} \left[\frac{f(a + h) - f(a)}{h} + \frac{g(a + h) - g(a)}{h} \right].$$

Use the limit law for sums:

$$(f + g)'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} + \lim_{h \rightarrow 0} \frac{g(a + h) - g(a)}{h}.$$

Since f and g are differentiable at a , both limits exist and equal $f'(a)$ and $g'(a)$, respectively. Therefore,

$$(f + g)'(a) = f'(a) + g'(a).$$

Question 2

Assume that f is differentiable at $x = a$ and that c is a constant. Prove that

$$(cf)'(a) = c f'(a).$$

Question 2 - Solution

By definition of the derivative,

$$(cf)'(a) = \lim_{h \rightarrow 0} \frac{cf(a+h) - cf(a)}{h}.$$

Factor out the constant c from the numerator:

$$(cf)'(a) = \lim_{h \rightarrow 0} \frac{c(f(a+h) - f(a))}{h}.$$

Rewrite the fraction:

$$(cf)'(a) = c \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Since f is differentiable at a , the limit exists and equals $f'(a)$. Therefore,

$$(cf)'(a) = c f'(a).$$

Question 3

Assume that f and g are differentiable at $x = a$. Prove the product rule:

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Question 3 - Solution

By definition of the derivative,

$$(fg)'(a) = \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}.$$

Add and subtract $f(a+h)g(a)$ in the numerator:

$$= \lim_{h \rightarrow 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}.$$

Group the terms:

$$= \lim_{h \rightarrow 0} \left[\frac{f(a+h)(g(a+h) - g(a))}{h} + \frac{g(a)(f(a+h) - f(a))}{h} \right].$$

Separate the limit:

$$= \lim_{h \rightarrow 0} f(a+h) \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} + g(a) \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Since f is differentiable at a , it is continuous at a , so

$$\lim_{h \rightarrow 0} f(a+h) = f(a).$$

Also, by differentiability,

$$\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} = g'(a), \quad \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Substitute these limits:

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a).$$

Thus,

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Question 4

Assume that f and g are differentiable at $x = a$ and that $g(a) \neq 0$. Prove the quotient rule:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Question 4 - Solution

By definition of the derivative,

$$\left(\frac{f}{g}\right)'(a) = \lim_{h \rightarrow 0} \frac{\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}}{h}.$$

Combine the fractions in the numerator:

$$= \lim_{h \rightarrow 0} \frac{f(a+h)g(a) - f(a)g(a+h)}{h g(a+h)g(a)}.$$

Rewrite the numerator by adding and subtracting $f(a)g(a)$:

$$f(a+h)g(a) - f(a)g(a+h) = g(a)(f(a+h) - f(a)) - f(a)(g(a+h) - g(a)).$$

Substitute this expression:

$$= \lim_{h \rightarrow 0} \frac{g(a)(f(a+h) - f(a)) - f(a)(g(a+h) - g(a))}{h g(a+h)g(a)}.$$

Separate the fraction:

$$= \lim_{h \rightarrow 0} \left[\frac{g(a)}{g(a+h)g(a)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a+h)g(a)} \cdot \frac{g(a+h) - g(a)}{h} \right].$$

Simplify coefficients:

$$= \lim_{h \rightarrow 0} \left[\frac{1}{g(a+h)} \cdot \frac{f(a+h) - f(a)}{h} - \frac{f(a)}{g(a)g(a+h)} \cdot \frac{g(a+h) - g(a)}{h} \right].$$

Since g is differentiable at a , it is continuous at a , so

$$\lim_{h \rightarrow 0} g(a+h) = g(a).$$

Taking limits term by term,

$$\left(\frac{f}{g}\right)'(a) = \frac{1}{g(a)}f'(a) - \frac{f(a)}{[g(a)]^2}g'(a).$$

Combine the terms:

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Question 4

Assume that f is differentiable at $x = a$. Prove that f is continuous at $x = a$.

Question 4 - Solution

To show that f is continuous at $x = a$, we must prove that

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Since f is differentiable at a , by definition,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Because this limit exists, the expression

$$\frac{f(a+h) - f(a)}{h}$$

remains bounded and approaches a finite value as $h \rightarrow 0$.

Rewrite $f(a+h) - f(a)$ as

$$f(a+h) - f(a) = \left(\frac{f(a+h) - f(a)}{h} \right) h.$$

Now take limits as $h \rightarrow 0$:

$$\lim_{h \rightarrow 0} (f(a+h) - f(a)) = \left(\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \right) \left(\lim_{h \rightarrow 0} h \right).$$

Since

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a) \quad \text{and} \quad \lim_{h \rightarrow 0} h = 0,$$

we obtain

$$\lim_{h \rightarrow 0} (f(a+h) - f(a)) = 0.$$

Therefore,

$$\lim_{h \rightarrow 0} f(a+h) = f(a).$$

This shows that

$$\lim_{x \rightarrow a} f(x) = f(a),$$

so f is continuous at $x = a$.

Question 6

Assume that f is differentiable at $x = a$ and that $f'(a) = 0$. Prove that

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = 0.$$

Question 6 - Solution

By the definition of the derivative,

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

We are given that

$$f'(a) = 0.$$

Thus,

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = 0.$$

To make this explicit using ε - δ language, let $\varepsilon > 0$ be given.

Since the limit defining $f'(a)$ exists and equals 0, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| < \varepsilon.$$

That is,

$$\left| \frac{f(x) - f(a)}{x - a} \right| < \varepsilon.$$

This proves that

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = 0.$$

Question 7

Prove that the derivative of a constant function is zero. That is, if

$$f(x) = c$$

for all x , where c is a constant, then

$$f'(a) = 0$$

for every real number a .

Question 7 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Since $f(x) = c$ for all x , we have

$$f(a+h) = c \quad \text{and} \quad f(a) = c.$$

Substitute into the difference quotient:

$$f'(a) = \lim_{h \rightarrow 0} \frac{c - c}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \rightarrow 0} \frac{0}{h}.$$

Since $\frac{0}{h} = 0$ for all $h \neq 0$, the limit is

$$f'(a) = 0.$$

Therefore, the derivative of a constant function is zero:

$$\boxed{f'(a) = 0}.$$

Question 8

Prove from first principles that if

$$f(x) = x^2,$$

then

$$f'(a) = 2a.$$

Question 8 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Substitute $f(x) = x^2$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h}.$$

Expand the square:

$$(a+h)^2 = a^2 + 2ah + h^2.$$

Substitute back:

$$f'(a) = \lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - a^2}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \rightarrow 0} \frac{2ah + h^2}{h}.$$

Factor out h :

$$f'(a) = \lim_{h \rightarrow 0} \frac{h(2a + h)}{h}.$$

Cancel h (for $h \neq 0$):

$$f'(a) = \lim_{h \rightarrow 0} (2a + h).$$

Now take the limit:

$$f'(a) = 2a.$$

Thus, using first principles,

$$\boxed{f'(a) = 2a}.$$

Question 9

Prove from first principles that if

$$f(x) = x^3,$$

then

$$f'(a) = 3a^2.$$

Question 9 - Solution

By definition of the derivative,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Substitute $f(x) = x^3$:

$$f'(a) = \lim_{h \rightarrow 0} \frac{(a+h)^3 - a^3}{h}.$$

Expand the cube:

$$(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3.$$

Substitute into the difference quotient:

$$f'(a) = \lim_{h \rightarrow 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}.$$

Simplify the numerator:

$$f'(a) = \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3}{h}.$$

Factor out h :

$$f'(a) = \lim_{h \rightarrow 0} \frac{h(3a^2 + 3ah + h^2)}{h}.$$

Cancel h (for $h \neq 0$):

$$f'(a) = \lim_{h \rightarrow 0} (3a^2 + 3ah + h^2).$$

Now take the limit:

$$f'(a) = 3a^2.$$

Thus, using first principles,

$$\boxed{f'(a) = 3a^2}.$$

Question 10

Assume that f is differentiable at $x = a$ and that g is differentiable at $x = f(a)$. Prove the Chain Rule:

$$(g \circ f)'(a) = g'(f(a)) f'(a).$$

Question 10 - Solution

Put $b = f(a)$. Differentiability of g at b means

$$g(b + u) - g(b) = g'(b)u + u\eta(u),$$

where $\eta(u) \rightarrow 0$ as $u \rightarrow 0$; define $\eta(0) = 0$ so this identity also holds at $u = 0$.

For $h \neq 0$, set $u = f(a + h) - f(a)$. Then

$$\frac{g(f(a + h)) - g(f(a))}{h} = (g'(b) + \eta(f(a + h) - f(a))) \frac{f(a + h) - f(a)}{h}.$$

Differentiability implies continuity of f at a , so the first factor tends to $g'(b)$ and the second to $f'(a)$.

$$\boxed{(g \circ f)'(a) = g'(f(a))f'(a).}$$

This argument remains valid when $f(a + h) = f(a)$ for some or all nearby h .