

Question 1

Assume $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Prove that

$$\lim_{x \rightarrow a} (f(x)g(x)) = LM.$$

Question 1 - Solution

Let $\varepsilon > 0$ be given.

We must show there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|f(x)g(x) - LM| < \varepsilon.$$

Start by adding and subtracting $Lg(x)$:

$$f(x)g(x) - LM = f(x)g(x) - Lg(x) + Lg(x) - LM = g(x)(f(x) - L) + L(g(x) - M).$$

Take absolute values and use the triangle inequality:

$$|f(x)g(x) - LM| \leq |g(x)| |f(x) - L| + |L| |g(x) - M|.$$

To control $|g(x)|$, use the fact that $g(x) \rightarrow M$.

Since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then

$$|g(x) - M| < 1.$$

Then, for $0 < |x - a| < \delta_1$,

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|.$$

Now choose $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}.$$

This is possible because $\lim_{x \rightarrow a} f(x) = L$.

Next choose $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then

$$|g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

This is possible because $\lim_{x \rightarrow a} g(x) = M$.

Let

$$\delta = \min(\delta_1, \delta_2, \delta_3).$$

Then whenever $0 < |x - a| < \delta$, we have simultaneously:

$$|g(x)| < 1 + |M|, \quad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}, \quad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

Substitute into the estimate:

$$|f(x)g(x) - LM| \leq |g(x)| |f(x) - L| + |L| |g(x) - M|.$$

For the first term:

$$|g(x)| |f(x) - L| < (1 + |M|) \cdot \frac{\varepsilon}{2(1 + |M|)} = \frac{\varepsilon}{2}.$$

For the second term:

$$|L| |g(x) - M| \leq \frac{\varepsilon |L|}{2(1 + |L|)} < \frac{\varepsilon}{2}.$$

Therefore,

$$|f(x)g(x) - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{x \rightarrow a} (f(x)g(x)) = LM.$$

Question 2

Assume that $\lim_{x \rightarrow a} f(x) = L$ and that c is a constant. Prove that

$$\lim_{x \rightarrow a} (cf(x)) = cL.$$

Question 2 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|cf(x) - cL| < \varepsilon.$$

Start by factoring out the constant:

$$|cf(x) - cL| = |c| \cdot |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \frac{\varepsilon}{|c|},$$

provided $c \neq 0$.

Substitute this into the inequality:

$$|cf(x) - cL| = |c| \cdot |f(x) - L| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$, we have

$$|cf(x) - cL| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (cf(x)) = cL.$$

If $c = 0$, then $cf(x) = 0$ for all x , and the result holds trivially.

Question 3

Assume that $\lim_{x \rightarrow a} f(x) = L$. Prove that

$$\lim_{x \rightarrow a} |f(x)| = |L|.$$

Question 3 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$||f(x)| - |L|| < \varepsilon.$$

Recall the inequality

$$||u| - |v|| \leq |u - v|$$

for all real numbers u and v .

Apply this inequality with $u = f(x)$ and $v = L$:

$$||f(x)| - |L|| \leq |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Combining the inequalities gives

$$||f(x)| - |L|| \leq |f(x) - L| < \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$,

$$||f(x)| - |L|| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} |f(x)| = |L|.$$

Question 4

Assume that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, with $M \neq 0$. Prove that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Question 4 - Solution

Let $\varepsilon > 0$. Since $g(x) \rightarrow M \neq 0$, choose $\delta_1 > 0$ such that $|g(x) - M| < |M|/2$ when $0 < |x - a| < \delta_1$. Then $|g(x)| > |M|/2$.

By the two assumed limits, choose $\delta_2, \delta_3 > 0$ to ensure respectively

$$|f(x) - L| < \frac{\varepsilon|M|}{4}, \quad |g(x) - M| < \frac{\varepsilon|M|^2}{4(1+|L|)}.$$

Take $\delta = \min(\delta_1, \delta_2, \delta_3)$. For $0 < |x - a| < \delta$,

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &\leq \frac{|f(x) - L|}{|g(x)|} + \frac{|L| |g(x) - M|}{|M| |g(x)|} \\ &\leq \frac{2|f(x) - L|}{|M|} + \frac{2|L| |g(x) - M|}{|M|^2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon|L|}{2(1+|L|)} < \varepsilon. \end{aligned}$$

(The first term makes the penultimate inequality strict even if $L = 0$.) This proves $\lim_{x \rightarrow a} f(x)/g(x) = L/M$.

Question 5

Assume that $\lim_{x \rightarrow a} f(x) = L$. Prove that

$$\lim_{x \rightarrow a} (f(x) + c) = L + c,$$

where c is a constant.

Question 5 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|(f(x) + c) - (L + c)| < \varepsilon.$$

Simplify the expression inside the absolute value:

$$|(f(x) + c) - (L + c)| = |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, by definition of the limit there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Therefore, for the same δ ,

$$|(f(x) + c) - (L + c)| < \varepsilon.$$

This proves that

$$\lim_{x \rightarrow a} (f(x) + c) = L + c.$$

Question 6

Assume that $\lim_{x \rightarrow a} f(x) = L$. Prove that

$$\lim_{x \rightarrow a} (f(x) - c) = L - c,$$

where c is a constant.

Question 6 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|(f(x) - c) - (L - c)| < \varepsilon.$$

Simplify the expression:

$$|(f(x) - c) - (L - c)| = |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, by the definition of the limit there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Thus, for this same choice of δ ,

$$|(f(x) - c) - (L - c)| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (f(x) - c) = L - c.$$

Question 7

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M,$$

and that

$$f(x) \leq g(x) \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \leq M.$$

Question 7 - Solution

We argue by contradiction.

Suppose, to the contrary, that

$$L > M.$$

Then the positive number

$$\varepsilon = \frac{L - M}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

This implies

$$f(x) > L - \varepsilon = \frac{L + M}{2}.$$

Similarly, since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|g(x) - M| < \varepsilon.$$

This implies

$$g(x) < M + \varepsilon = \frac{L + M}{2}.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$, we have

$$f(x) > \frac{L + M}{2} \quad \text{and} \quad g(x) < \frac{L + M}{2}.$$

Thus,

$$f(x) > g(x),$$

which contradicts the assumption that $f(x) \leq g(x)$ near a .

Therefore, our assumption that $L > M$ must be false, and we conclude that

$$L \leq M.$$

Question 8

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad f(x) \geq 0 \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \geq 0.$$

Question 8 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L < 0.$$

Then the positive number

$$\varepsilon = \frac{-L}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

This inequality implies

$$f(x) < L + \varepsilon = L + \frac{-L}{2} = \frac{L}{2}.$$

Because $L < 0$, we have

$$\frac{L}{2} < 0.$$

Thus, for all x with $0 < |x - a| < \delta$,

$$f(x) < 0.$$

This contradicts the assumption that $f(x) \geq 0$ for all x near a .

Therefore, our assumption that $L < 0$ must be false. We conclude that

$$L \geq 0.$$

Question 9

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = L.$$

Prove that

$$\lim_{x \rightarrow a} (f(x) - g(x)) = 0.$$

Question 9 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - g(x)| < \varepsilon.$$

Begin by rewriting the difference:

$$f(x) - g(x) = (f(x) - L) - (g(x) - L).$$

Take absolute values and apply the triangle inequality:

$$|f(x) - g(x)| \leq |f(x) - L| + |g(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_1.$$

Similarly, since $\lim_{x \rightarrow a} g(x) = L$, there exists $\delta_2 > 0$ such that

$$|g(x) - L| < \frac{\varepsilon}{2} \quad \text{whenever } 0 < |x - a| < \delta_2.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$,

$$|f(x) - g(x)| \leq |f(x) - L| + |g(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (f(x) - g(x)) = 0.$$

Question 10

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} f(x) = M.$$

Prove that

$$L = M.$$

Question 10 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L \neq M.$$

Then the positive number

$$\varepsilon = \frac{|L - M|}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

Similarly, since $\lim_{x \rightarrow a} f(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|f(x) - M| < \varepsilon.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x such that $0 < |x - a| < \delta$, both inequalities hold.

Using the triangle inequality,

$$|L - M| = |L - f(x) + f(x) - M| \leq |f(x) - L| + |f(x) - M|.$$

Substitute the bounds:

$$|L - M| < \varepsilon + \varepsilon = 2\varepsilon = |L - M|.$$

This is a contradiction.

Therefore, our assumption that $L \neq M$ is false. We conclude that

$$L = M.$$