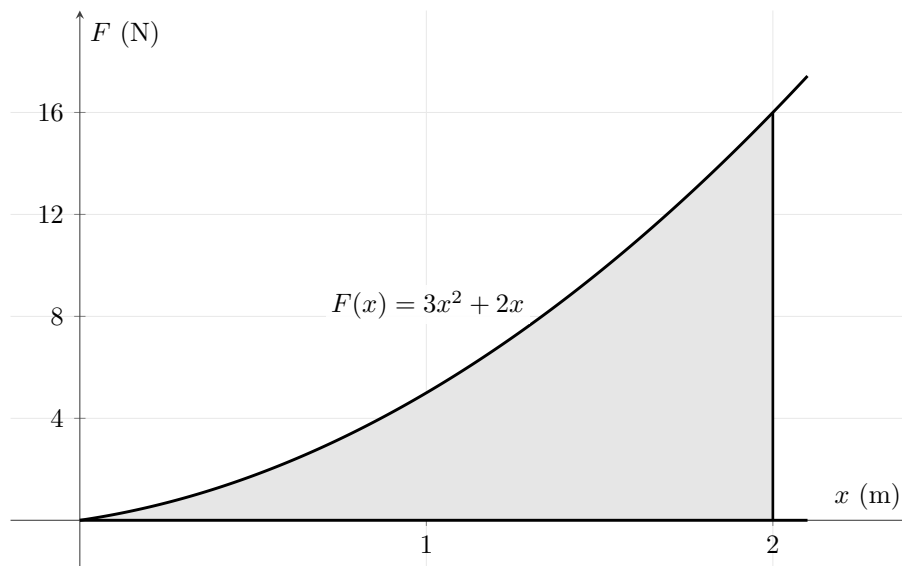


Question 1

A force acting along a straight line is given by

$$F(x) = 3x^2 + 2x \quad (\text{newtons}),$$

where x is the displacement in meters from the origin. Find the work done by the force in moving an object from $x = 0$ to $x = 2$.



The shaded area represents work over the specified interval.

Question 1 – Solution

1. Identify the force and interval.

The force acts along the displacement and is $F(x) = 3x^2 + 2x$ newtons. The object moves from $x = 0$ to $x = 2$ meters.

2. Write the work integral.

Over a small displacement dx , $dW = F(x) dx$. Therefore,

$$W = \int_0^2 F(x) dx = \int_0^2 (3x^2 + 2x) dx.$$

3. Integrate term by term.

By the power rule,

$$\int 3x^2 dx = 3\frac{x^3}{3} = x^3, \quad \int 2x dx = 2\frac{x^2}{2} = x^2.$$

4. Apply the bounds.

$$W = [x^3 + x^2]_0^2 = (2^3 + 2^2) - (0^3 + 0^2) = 8 + 4 = 12.$$

5. State the result with units.

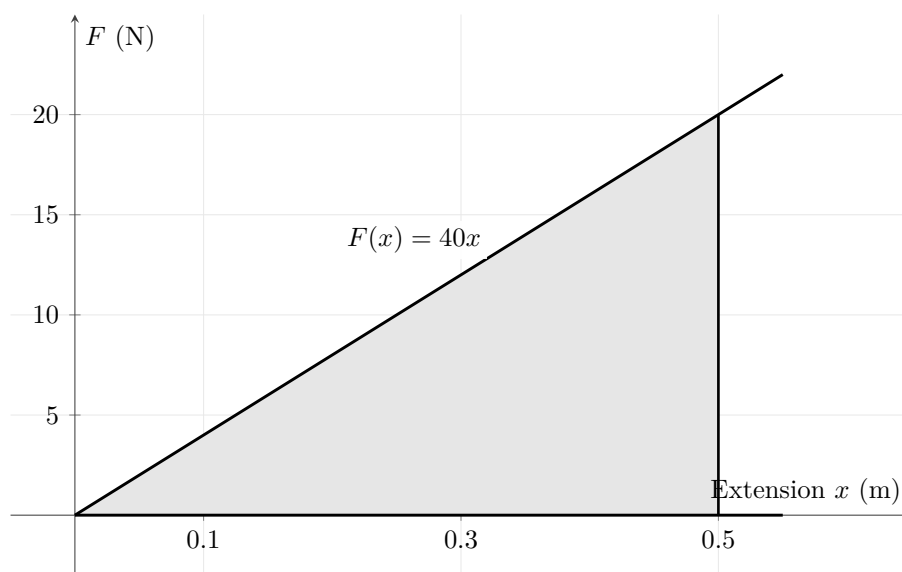
$$\boxed{W = 12 \text{ J}}.$$

A newton times a meter is a joule. Since $F(x) \geq 0$ on $[0, 2]$, the work is the positive shaded area under the force curve.

Question 2

A spring has natural length 0.5 m. A force of 12 N is required to stretch the spring to a length of 0.8 m.

Find the work required to stretch the spring from its natural length to a length of 1.0 m.



The shaded area represents work over the specified interval.

Question 2 – Solution

1. Use extension beyond natural length.

Let x be extension, not total spring length. The calibration force produces

$$x = 0.8 - 0.5 = 0.3 \text{ m.}$$

2. Find the spring constant.

Hooke's law for the required stretching force is $F = kx$:

$$12 = k(0.3) \implies k = \frac{12}{0.3} = 40 \text{ N/m.}$$

3. Translate the requested lengths into bounds.

The initial and final extensions are

$$x_0 = 0.5 - 0.5 = 0, \quad x_1 = 1.0 - 0.5 = 0.5 \text{ m.}$$

Thus $F(x) = 40x$ and $W = \int_0^{0.5} 40x \, dx$.

4. Integrate and evaluate.

$$W = [20x^2]_0^{0.5} = 20(0.5)^2 - 0 = 20(0.25) = 5.$$

5. State and check the work.

$$\boxed{W = 5 \text{ J}}.$$

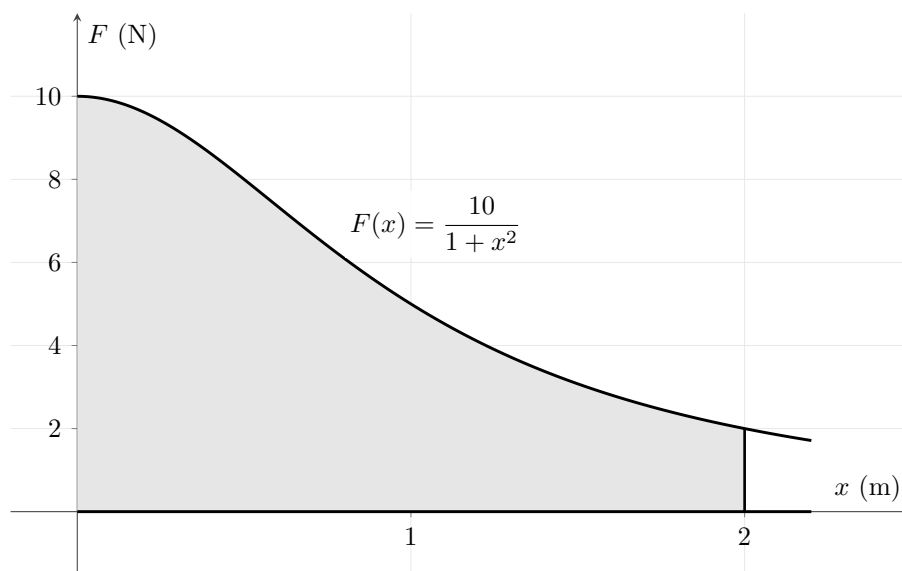
The force rises linearly from 0 to 20 N, so the triangular area is also $\frac{1}{2}(0.5)(20) = 5 \text{ J}$.

Question 3

A force acting on an object is given by

$$F(x) = \frac{10}{1+x^2} \quad (\text{newtons}),$$

where x is the distance (in meters) the object has moved along a straight line. Find the work done in moving the object from $x = 0$ to $x = 2$.



The shaded area represents work over the specified interval.

Question 3 – Solution

1. Identify the work model.

The force is positive over $0 \leq x \leq 2$, so the work equals the area under the force curve:

$$W = \int_0^2 F(x) dx.$$

2. Substitute the force.

$$W = \int_0^2 \frac{10}{1+x^2} dx = 10 \int_0^2 \frac{1}{1+x^2} dx.$$

3. Use the inverse-tangent antiderivative.

Since $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$,

$$W = 10 [\arctan x]_0^2.$$

Inverse-trigonometric values are in radians.

4. Apply the endpoints.

$$W = 10(\arctan 2 - \arctan 0) = 10 \arctan 2.$$

5. State the exact and approximate work.

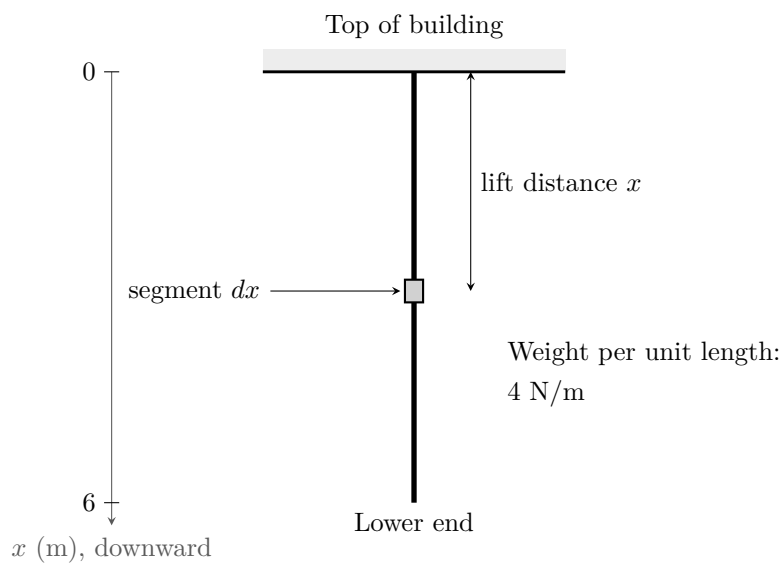
$$\boxed{W = 10 \arctan 2 \text{ J}} \approx 11.0715 \text{ J}.$$

The force decreases from 10 N to 2 N; the resulting work lies between $2(2) = 4 \text{ J}$ and $10(2) = 20 \text{ J}$.

Question 4

A 6-meter cable hangs vertically from the top of a building. The cable has constant linear weight density 4 N/m .

Find the work required to lift the entire cable to the top of the building.



Initial configuration; x is measured down from the top.

Question 4 – Solution

1. Choose a coordinate for each cable segment.

Let x measure the initial distance downward from the top of the building. The cable occupies $0 \leq x \leq 6$ meters.

2. Find the weight of a small segment.

The given quantity is weight per unit length. A segment of length dx weighs

$$dF = (4 \text{ N/m}) dx.$$

No additional factor of gravity is needed.

3. Find its lifting distance and work.

The segment initially at depth x must rise x meters. Thus

$$dW = (\text{weight})(\text{distance}) = 4x dx.$$

4. Add the work for all segments.

$$W = \int_0^6 4x dx = [2x^2]_0^6 = 2(36) - 0 = 72.$$

5. State and check the result.

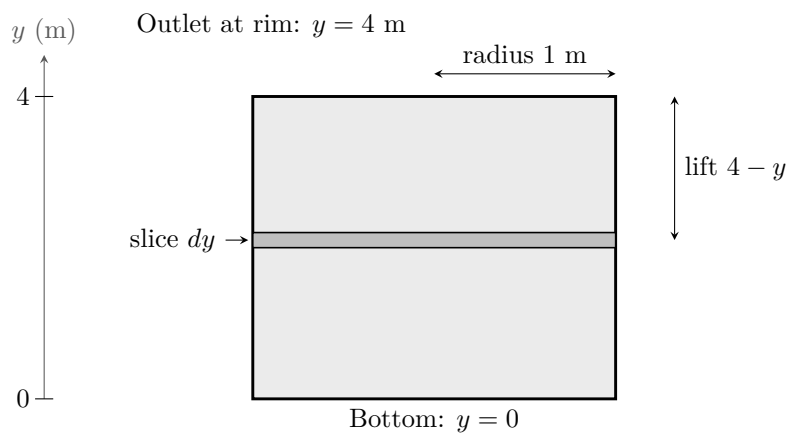
$$\boxed{W = 72 \text{ J}}.$$

The cable weighs $4(6) = 24$ N, and its center of mass rises 3 m. This gives the same work: $24(3) = 72$ J.

Question 5

A cylindrical tank of radius 1 m and height 4 m is full of water. Water weighs 9800 N/m^3 .

Find the work required to pump all the water to the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

Question 5 – Solution

1. Choose horizontal slices.

Let y be height above the tank bottom, with $0 \leq y \leq 4$. A slice at height y has thickness dy .

2. Find the slice volume and weight.

The horizontal circular area is $A = \pi(1)^2 = \pi \text{ m}^2$. Hence

$$dV = \pi dy, \quad dF = 9800 dV = 9800\pi dy.$$

The density is already a weight density.

3. Find the distance to the outlet.

The outlet is at the top, $y = 4$, so the slice must rise $4 - y$ meters:

$$dW = 9800\pi(4 - y) dy.$$

4. Set up and evaluate the work integral.

$$W = 9800\pi \int_0^4 (4 - y) dy = 9800\pi \left[4y - \frac{y^2}{2} \right]_0^4.$$
$$W = 9800\pi(16 - 8) = 78400\pi.$$

5. State and check the work.

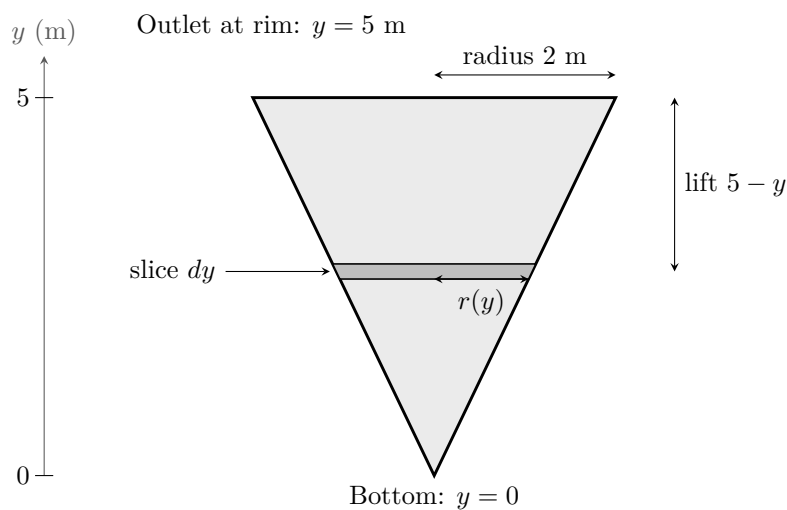
$$\boxed{W = 78400\pi \text{ J}}.$$

The total water weight is $9800(4\pi) = 39200\pi \text{ N}$. Its average lifting distance is 2 m, giving $78400\pi \text{ J}$.

Question 6

A conical tank with its vertex at the bottom, height 5 m, and top radius 2 m is full of water. Water weighs 9800 N/m^3 .

Find the work required to pump all the water to the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

Question 6 – Solution

1. Choose a height coordinate and determine the radius.

Measure y upward from the bottom vertex, so $0 \leq y \leq 5$. Similar triangles give

$$\frac{r(y)}{y} = \frac{2}{5} \implies r(y) = \frac{2y}{5}.$$

2. Find the volume and weight of a slice.

$$dV = \pi r(y)^2 dy = \frac{4\pi}{25} y^2 dy, \quad dF = 9800 dV = \frac{39200\pi}{25} y^2 dy.$$

3. Include the lifting distance.

The outlet is at $y = 5$, so a slice at y rises $5 - y$ meters:

$$W = \frac{39200\pi}{25} \int_0^5 y^2 (5 - y) dy.$$

4. Expand and integrate.

$$y^2(5 - y) = 5y^2 - y^3, \quad W = \frac{39200\pi}{25} \left[\frac{5}{3} y^3 - \frac{y^4}{4} \right]_0^5.$$

5. Apply the bounds and simplify.

$$\begin{aligned} W &= \frac{39200\pi}{25} \left(\frac{625}{3} - \frac{625}{4} \right) \\ &= \frac{39200\pi}{25} \cdot \frac{625}{12} = \boxed{\frac{245000\pi}{3} \text{ J}}. \end{aligned}$$

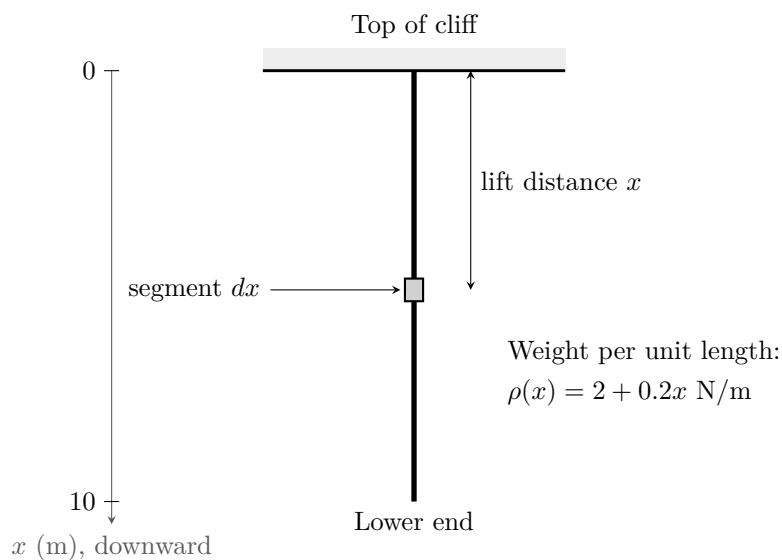
Question 7

A 10-meter rope hangs vertically from the edge of a cliff. The rope has variable linear weight density given by

$$\rho(x) = 2 + 0.2x \quad \text{N/m},$$

where x is the distance (in meters) measured downward from the top of the cliff.

Find the work required to pull the entire rope to the top of the cliff.



Initial configuration; x is measured down from the top.

Question 7 – Solution

1. Label each segment by its initial depth.

Let x be distance downward from the cliff top, as in the question. The rope occupies $0 \leq x \leq 10$ meters; $\rho(x)$ describes the initial weight distribution.

2. Find the segment weight.

For a short piece of length dx ,

$$dF = \rho(x) dx = (2 + 0.2x) dx.$$

The units are newtons because ρ is given in N/m.

3. Multiply by the distance lifted.

The segment at depth x is lifted x meters:

$$dW = x(2 + 0.2x) dx, \quad W = \int_0^{10} x(2 + 0.2x) dx.$$

4. Expand and integrate.

$$W = \int_0^{10} \left(2x + \frac{x^2}{5} \right) dx = \left[x^2 + \frac{x^3}{15} \right]_0^{10}.$$

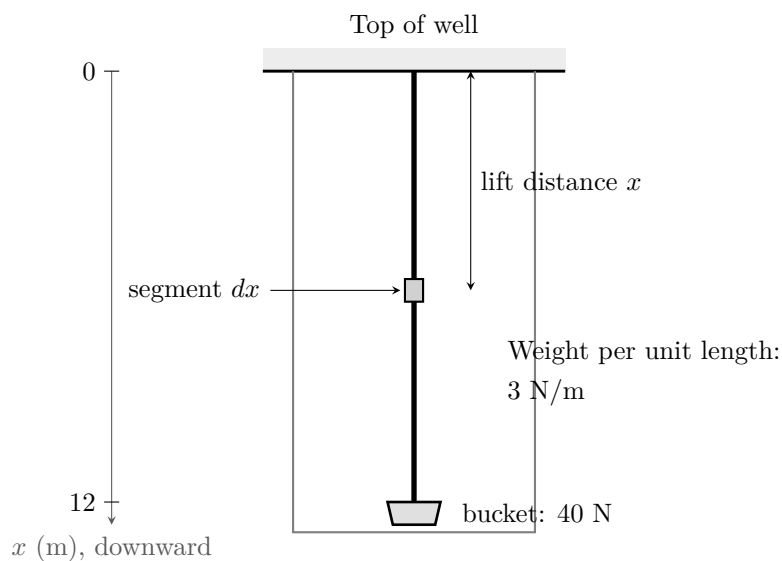
5. Evaluate and state the work.

$$W = 100 + \frac{1000}{15} = 100 + \frac{200}{3} = \boxed{\frac{500}{3} \text{ J}} \approx 166.667 \text{ J}.$$

Question 8

A bucket is lifted from the bottom of a well that is 12 m deep. The bucket itself weighs 40 N. Attached to the bucket is a rope that weighs 3 N/m.

Find the work required to lift the bucket to the top of the well.



Initial configuration; x is measured down from the top.

Question 8 – Solution

1. Separate the bucket and rope contributions.

The bucket has constant weight, while different pieces of rope rise different distances:

$$W_{\text{total}} = W_{\text{bucket}} + W_{\text{rope}}.$$

2. Calculate the bucket work.

The 40 N bucket rises the full 12 m:

$$W_{\text{bucket}} = 40(12) = 480 \text{ J}.$$

3. Define a consistent coordinate for the rope.

Let x be the initial depth below the top of the well, with $0 \leq x \leq 12$. A segment of length dx weighs $3 dx$ newtons and rises x meters.

4. Integrate the rope work.

$$W_{\text{rope}} = \int_0^{12} 3x \, dx = \left[\frac{3}{2}x^2 \right]_0^{12} = \frac{3}{2}(144) = 216 \text{ J}.$$

5. Add and check the total.

$$W_{\text{total}} = 480 + 216 = \boxed{696 \text{ J}}.$$

As a check, the rope weighs $3(12) = 36$ N and its center of mass rises 6 m, giving $36(6) = 216$ J.

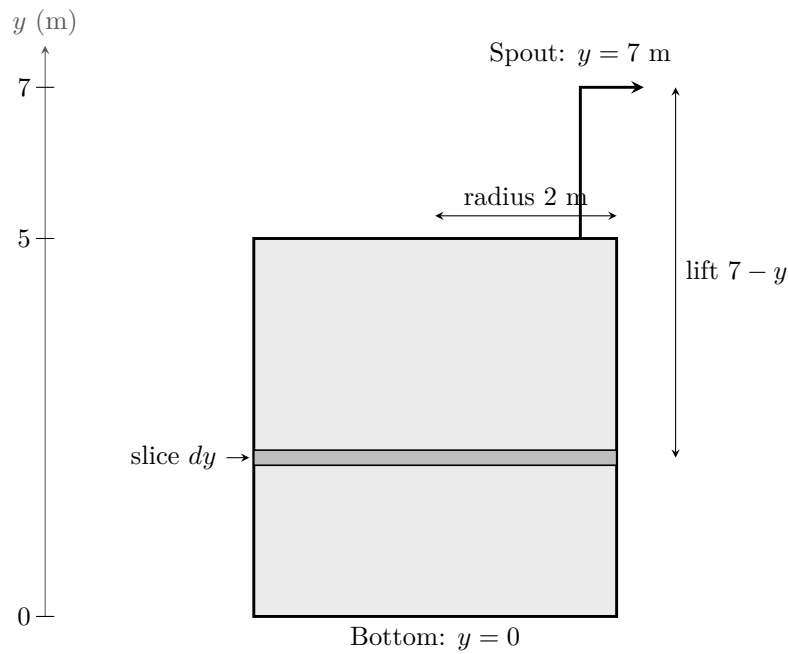
Question 9

A cylindrical tank of radius 2 m and height 5 m is full of oil. The oil has weight density

$$w(y) = 9000(1 + 0.1y) \text{ N/m}^3,$$

where y is the height (in meters) above the bottom of the tank.

Find the work required to pump all the oil to a spout located 2 m above the top of the tank.



Vertical section through the tank; the shaded band represents a horizontal slice.

Question 9 – Solution

1. Locate the slices and outlet.

Let y be height above the bottom. Oil occupies $0 \leq y \leq 5$, and the spout is at height $5 + 2 = 7$ m. The lift distance is $7 - y$.

2. Find the slice volume and weight.

The tank radius is 2 m, so

$$dV = \pi(2)^2 dy = 4\pi dy.$$

$$dF = w(y) dV = 9000(1 + 0.1y)(4\pi) dy = 36000\pi(1 + 0.1y) dy.$$

3. Set up the work integral.

Multiply each slice weight by its lifting distance:

$$W = 36000\pi \int_0^5 (1 + 0.1y)(7 - y) dy.$$

4. Expand and integrate exactly.

$$(1 + 0.1y)(7 - y) = 7 - \frac{3}{10}y - \frac{1}{10}y^2.$$

$$W = 36000\pi \left[7y - \frac{3}{20}y^2 - \frac{1}{30}y^3 \right]_0^5.$$

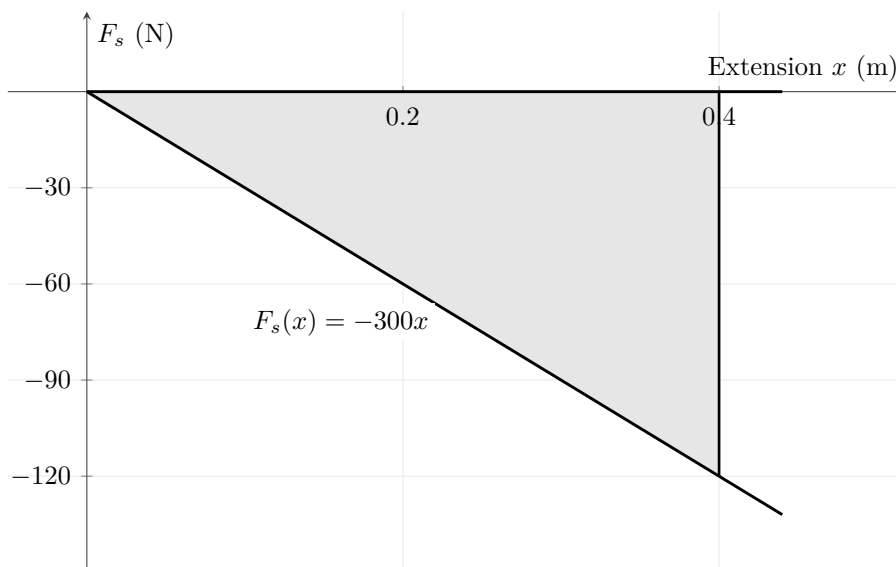
5. Apply the bounds and simplify.

$$\begin{aligned} W &= 36000\pi \left(35 - \frac{15}{4} - \frac{25}{6} \right) \\ &= 36000\pi \left(\frac{420 - 45 - 50}{12} \right) = \boxed{975000\pi \text{ J}}. \end{aligned}$$

Question 10

A vertical spring attached to a ceiling has natural length 1 m and spring constant $k = 300$ N/m. An external agent stretches it downward from length 1 m to 1.4 m.

Find the work done by the spring during this motion.



The shaded area represents negative work done by the spring.

Question 10 – Solution

1. Define displacement from natural length.

Let x be extension measured positively downward. The spring lengths correspond to

$$x_0 = 1 - 1 = 0, \quad x_1 = 1.4 - 1 = 0.4 \text{ m.}$$

2. Use the signed force exerted by the spring.

The spring pulls upward, opposite the positive direction. Therefore,

$$F_s(x) = -kx = -300x \text{ N.}$$

The question asks for work done by the spring itself.

3. Set up the signed work integral.

$$W_s = \int_0^{0.4} F_s(x) dx = \int_0^{0.4} -300x dx.$$

4. Integrate and apply the bounds.

$$W_s = [-150x^2]_0^{0.4} = -150(0.4)^2 = -150(0.16) = -24.$$

5. State the result and interpret its sign.

$$\boxed{W_s = -24 \text{ J.}}$$

The negative sign indicates that the spring force opposes the motion. Its elastic potential energy increases by $\frac{1}{2}k(0.4)^2 = 24 \text{ J}$, consistent with $W_s = -\Delta U$.