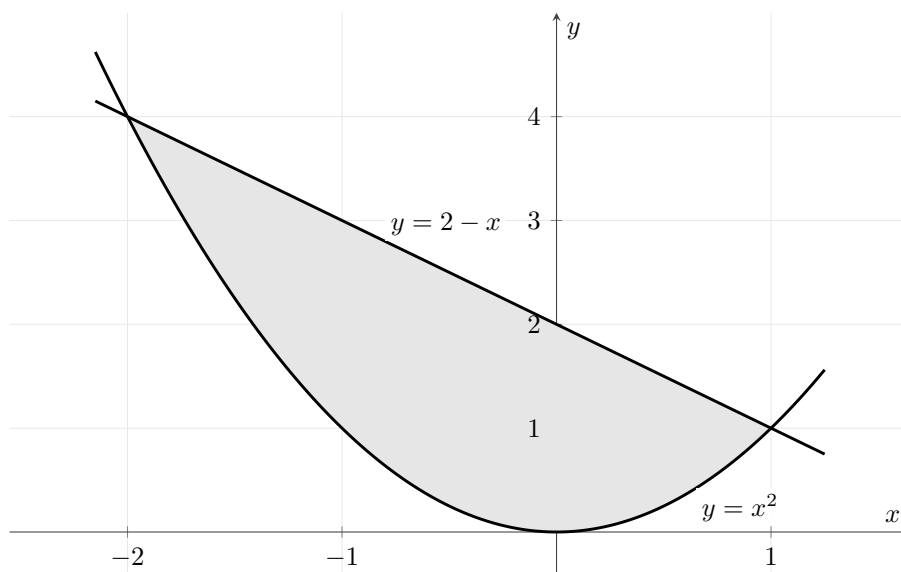


Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2, \quad y = 2 - x,$$

about the x -axis.



Rotate the shaded region about the x -axis.

Question 1 – Solution**1. Find the intersections.**

$$x^2 = 2 - x \implies x^2 + x - 2 = (x + 2)(x - 1) = 0.$$

Thus $x = -2$ and $x = 1$, giving intersection points $(-2, 4)$ and $(1, 1)$.

2. Choose washers and identify the radii.

Use vertical slices perpendicular to the x -axis. On $[-2, 1]$, $0 \leq x^2 \leq 2 - x$, so

$$R(x) = 2 - x, \quad r(x) = x^2.$$

3. Set up the volume and expand.

$$V = \pi \int_{-2}^1 (R^2 - r^2) dx = \pi \int_{-2}^1 ((2 - x)^2 - x^4) dx.$$
$$(2 - x)^2 - x^4 = 4 - 4x + x^2 - x^4.$$

4. Find an antiderivative.

Using the power rule for each term,

$$F(x) = 4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5}, \quad V = \pi[F(x)]_{-2}^1.$$

5. Evaluate the endpoints and subtract.

$$F(1) = 4 - 2 + \frac{1}{3} - \frac{1}{5} = \frac{32}{15}, \quad F(-2) = -8 - 8 - \frac{8}{3} + \frac{32}{5} = -\frac{184}{15}.$$

$$V = \pi \left(\frac{32}{15} + \frac{184}{15} \right) = \boxed{\frac{72\pi}{5}}.$$

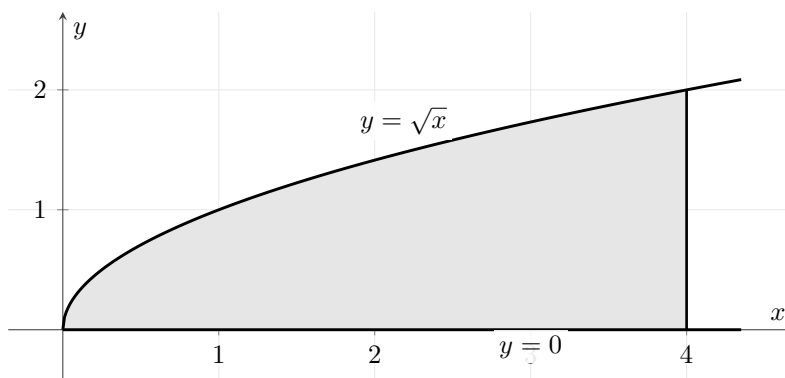
All volumes are in cubic units.

Question 2

The base of a solid is the region bounded by

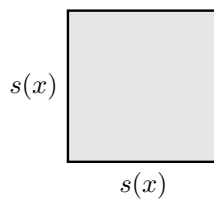
$$y = \sqrt{x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 4$. Cross-sections perpendicular to the x -axis are squares. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 2 – Solution**1. Choose slices and bounds.**

The cross-sections are perpendicular to the x -axis, so use slices of thickness dx on the given interval $0 \leq x \leq 4$.

2. Find the square side length.

The vertical segment in the base is the side of the square:

$$s(x) = y_{\text{top}} - y_{\text{bottom}} = \sqrt{x} - 0 = \sqrt{x}.$$

3. Write the cross-sectional area.

For a square, area is side squared:

$$A(x) = s(x)^2 = (\sqrt{x})^2 = x.$$

4. Set up and integrate the volume.

Add the areas of the thin cross-sections:

$$V = \int_0^4 A(x) \, dx = \int_0^4 x \, dx = \left[\frac{x^2}{2} \right]_0^4.$$

5. Evaluate both bounds.

$$V = \frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} - 0 = \boxed{8}.$$

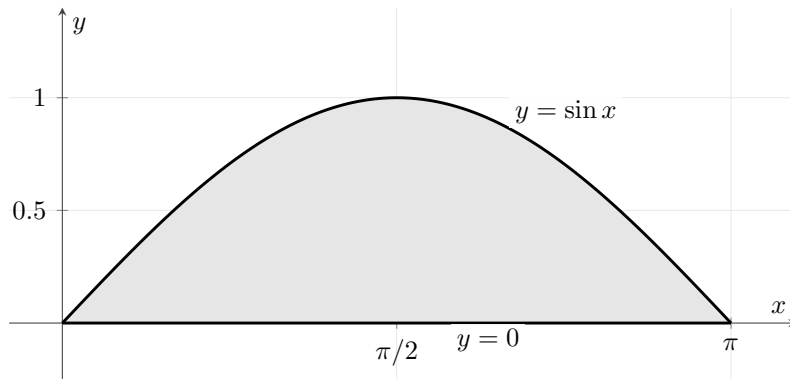
All volumes are in cubic units.

Question 3

The base of a solid is the region enclosed by

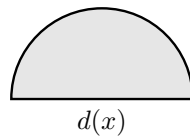
$$y = \sin x \quad \text{and} \quad y = 0$$

from $x = 0$ to $x = \pi$. Cross-sections perpendicular to the x -axis are semicircles whose diameters lie in the base region. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 3 – Solution**1. Identify the diameter and radius.**

Use dx slices for $0 \leq x \leq \pi$. The base segment is a diameter, so

$$d(x) = \sin x - 0 = \sin x, \quad r(x) = \frac{d(x)}{2} = \frac{\sin x}{2}.$$

2. Find the area of a semicircle.

$$A(x) = \frac{1}{2}\pi r(x)^2 = \frac{1}{2}\pi \left(\frac{\sin x}{2}\right)^2 = \frac{\pi}{8} \sin^2 x.$$

3. Set up the volume and use a trigonometric identity.

$$V = \frac{\pi}{8} \int_0^\pi \sin^2 x \, dx, \quad \sin^2 x = \frac{1 - \cos(2x)}{2}.$$
$$V = \frac{\pi}{16} \int_0^\pi (1 - \cos(2x)) \, dx.$$

4. Integrate with the chain-rule factor.

Since $\int \cos(2x) \, dx = \sin(2x)/2$,

$$V = \frac{\pi}{16} \left[x - \frac{\sin(2x)}{2} \right]_0^\pi.$$

5. Apply the bounds.

$$V = \frac{\pi}{16} \left[\left(\pi - \frac{\sin(2\pi)}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \right] = \boxed{\frac{\pi^2}{16}}.$$

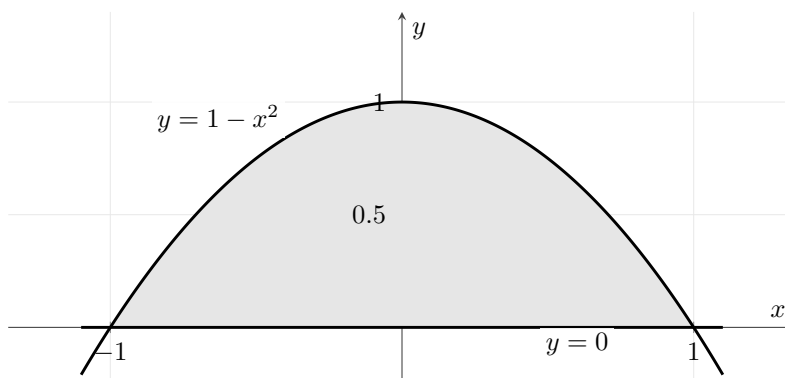
All volumes are in cubic units.

Question 4

The base of a solid is the region enclosed by

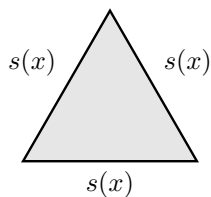
$$y = 1 - x^2 \quad \text{and} \quad y = 0$$

between their intersection points. Cross-sections perpendicular to the x -axis are equilateral triangles. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 4 – Solution**1. Find the bounds and side length.**

The intersections satisfy $1 - x^2 = 0$, so $x = -1, 1$. A vertical base segment gives

$$s(x) = 1 - x^2 \quad (-1 \leq x \leq 1).$$

2. Find the area of an equilateral triangle.

The altitude is $\sqrt{s^2 - (s/2)^2} = \sqrt{3} s/2$. Hence

$$A = \frac{1}{2}s \left(\frac{\sqrt{3}}{2}s \right) = \frac{\sqrt{3}}{4}s^2, \quad A(x) = \frac{\sqrt{3}}{4}(1 - x^2)^2.$$

3. Set up the volume and expand.

$$V = \frac{\sqrt{3}}{4} \int_{-1}^1 (1 - x^2)^2 dx = \frac{\sqrt{3}}{4} \int_{-1}^1 (1 - 2x^2 + x^4) dx.$$

4. Use symmetry and integrate.

The integrand is even, so

$$V = \frac{\sqrt{3}}{2} \int_0^1 (1 - 2x^2 + x^4) dx = \frac{\sqrt{3}}{2} \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1.$$

5. Evaluate and simplify.

$$V = \frac{\sqrt{3}}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{\sqrt{3}}{2} \left(\frac{15 - 10 + 3}{15} \right) = \boxed{\frac{4\sqrt{3}}{15}}.$$

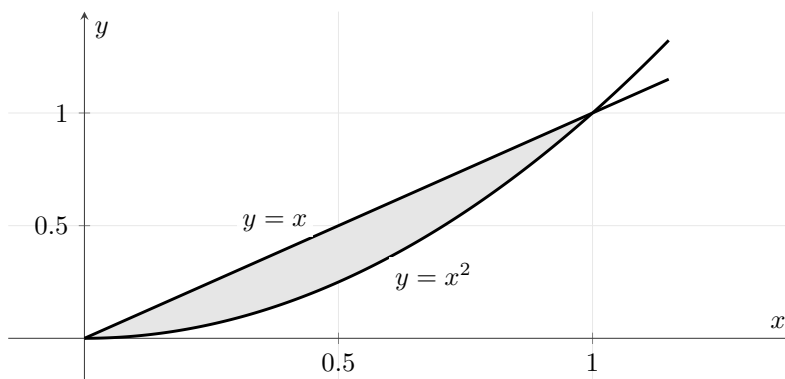
All volumes are in cubic units.

Question 5

The base of a solid is the region bounded by

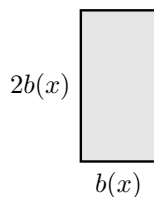
$$y = x \quad \text{and} \quad y = x^2$$

between their intersection points. Cross-sections perpendicular to the x -axis are rectangles whose height is *twice* the base in the region. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 5 – Solution

1. Find the interval and upper curve.

$$x = x^2 \implies x(x - 1) = 0 \implies x = 0, 1.$$

On $[0, 1]$, $x \geq x^2$, so the line lies above the parabola.

2. Identify the rectangle dimensions.

The base is the vertical distance between curves; its height is twice that distance:

$$b(x) = x - x^2, \quad h(x) = 2b(x) = 2(x - x^2).$$

3. Find the area and expand.

$$A(x) = b(x)h(x) = 2(x - x^2)^2 = 2x^2 - 4x^3 + 2x^4.$$

4. Set up and integrate the volume.

$$V = \int_0^1 A(x) dx = \left[\frac{2}{3}x^3 - x^4 + \frac{2}{5}x^5 \right]_0^1.$$

Each term follows from $\int x^n dx = x^{n+1}/(n+1)$.

5. Evaluate and combine the fractions.

$$V = \left(\frac{2}{3} - 1 + \frac{2}{5} \right) - 0 = \frac{10 - 15 + 6}{15} = \boxed{\frac{1}{15}}.$$

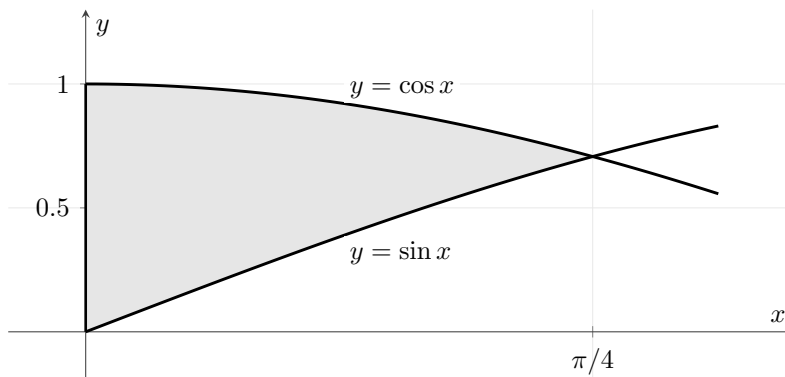
All volumes are in cubic units.

Question 6

The base of a solid is the region bounded by

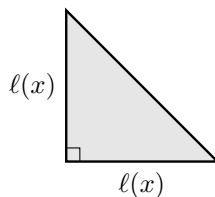
$$y = \cos x \quad \text{and} \quad y = \sin x$$

on the interval where $\cos x \geq \sin x$ in the first quadrant. Cross-sections perpendicular to the x -axis are isosceles right triangles with one leg lying in the base region. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 6 – Solution**1. Determine the bounds.**

In the first quadrant, $\cos x = \sin x$ at $x = \pi/4$. The specified order $\cos x \geq \sin x$ holds for $0 \leq x \leq \pi/4$.

2. Identify the triangle leg and area.

The segment in the base is a leg, not the hypotenuse. Both perpendicular legs have length

$$\ell(x) = \cos x - \sin x, \quad A(x) = \frac{1}{2}\ell(x)^2.$$

3. Set up the volume and simplify.

$$V = \frac{1}{2} \int_0^{\pi/4} (\cos x - \sin x)^2 dx.$$
$$(\cos x - \sin x)^2 = \cos^2 x + \sin^2 x - 2 \sin x \cos x = 1 - \sin(2x).$$

4. Integrate each term.

$$V = \frac{1}{2} \int_0^{\pi/4} (1 - \sin(2x)) dx = \frac{1}{2} \left[x + \frac{1}{2} \cos(2x) \right]_0^{\pi/4}.$$

5. Apply the bounds.

$$V = \frac{1}{2} \left[\left(\frac{\pi}{4} + \frac{1}{2} \cos \frac{\pi}{2} \right) - \left(0 + \frac{1}{2} \cos 0 \right) \right] = \boxed{\frac{\pi}{8} - \frac{1}{4}}.$$

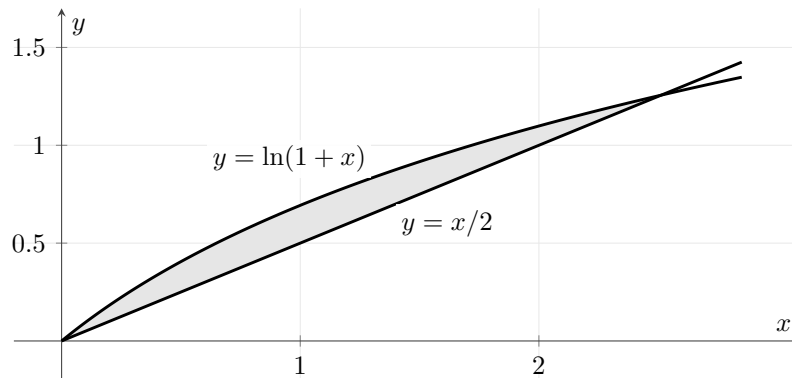
All volumes are in cubic units.

Question 7

The base of a solid is the region bounded by

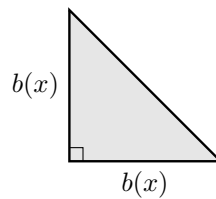
$$y = \ln(x + 1) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection. Cross-sections perpendicular to the x -axis are right triangles whose height is equal to the base. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 7 – Solution

1. Define the intersection bounds.

Let $b(x) = \ln(1+x) - x/2$. Since $b'(x) = (1-x)/(2(1+x))$, b increases to $x = 1$ and then decreases. Its zeros are 0 and a unique $a > 1$ with

$$\ln(1+a) = a/2, \quad a \approx 2.512862417.$$

This equation defines a exactly; $b(x) > 0$ on $(0, a)$.

2. Find the triangular area and set up the volume.

Both perpendicular legs have length $b(x)$, so

$$V = \frac{1}{2} \int_0^a \left(\ln(1+x) - \frac{x}{2} \right)^2 dx.$$

3. Expand and find the needed antiderivatives.

Write $L = \ln(1+x)$ and $u = 1+x$. The square is $L^2 - xL + x^2/4$. Integration by parts gives

$$\int L^2 dx = u(L^2 - 2L + 2), \quad \int xL dx = \frac{x^2}{2}L - \frac{x^2}{4} + \frac{x}{2} - \frac{L}{2}.$$

For the first identity, use $\int \ln^2 u du = u \ln^2 u - 2 \int \ln u du$.

4. Combine the terms and apply the endpoints.

An antiderivative of the squared expression is

$$H(x) = uL^2 - \left(u + \frac{u^2}{2}\right)L + u + \frac{u^2}{4} + \frac{x^3}{12}.$$

At $x = 0$, $H(0) = 5/4$. Substituting $u = 1+a$ and $L = a/2$ gives

$$H(a) - H(0) = \frac{a^3}{12} - \frac{a^2}{2} + \frac{3a}{4} = \frac{a(a-3)^2}{12}.$$

5. Include the triangular-area factor.

$$V = \frac{1}{2}[H(a) - H(0)] = \boxed{\frac{a(a-3)^2}{24}} \approx 0.024846244.$$

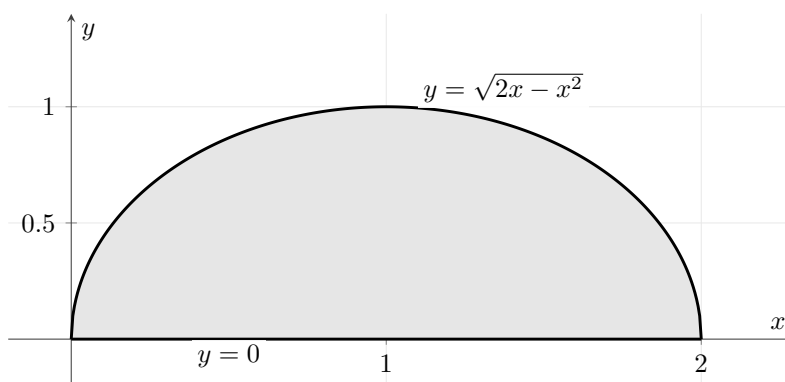
All volumes are in cubic units.

Question 8

The base of a solid is the region bounded by

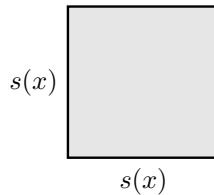
$$y = \sqrt{2x - x^2} \quad \text{and} \quad y = 0.$$

Cross-sections perpendicular to the x -axis are squares. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 8 – Solution**1. Determine the bounds.**

The square root is real when $2x - x^2 = x(2 - x) \geq 0$, giving $0 \leq x \leq 2$. These endpoints are also the intersections with $y = 0$.

2. Find the square side length.

The cross-sections are perpendicular to the x -axis. Their side length is the vertical segment in the base:

$$s(x) = \sqrt{2x - x^2} - 0.$$

3. Find the cross-sectional area.

$$A(x) = s(x)^2 = \left(\sqrt{2x - x^2}\right)^2 = 2x - x^2.$$

4. Set up and integrate the volume.

$$V = \int_0^2 A(x) dx = \int_0^2 (2x - x^2) dx = \left[x^2 - \frac{x^3}{3} \right]_0^2.$$

5. Evaluate and simplify.

$$V = \left(4 - \frac{8}{3} \right) - 0 = \frac{12 - 8}{3} = \boxed{\frac{4}{3}}.$$

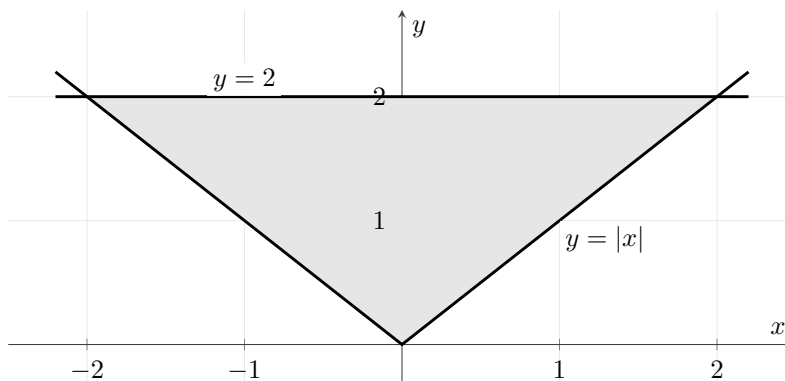
All volumes are in cubic units.

Question 9

The base of a solid is the region bounded by

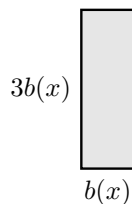
$$y = |x| \quad \text{and} \quad y = 2,$$

between their intersection points. Cross-sections perpendicular to the x -axis are rectangles whose height is three times their base. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 9 – Solution**1. Find the bounds.**

The intersections satisfy $|x| = 2$, giving $x = -2$ and $x = 2$. Use dx slices because the cross-sections are perpendicular to the x -axis.

2. Identify the rectangle dimensions.

The base length is top minus bottom, and the height is three times the base:

$$b(x) = 2 - |x|, \quad h(x) = 3(2 - |x|).$$

3. Set up the area and volume.

$$A(x) = b(x)h(x) = 3(2 - |x|)^2, \quad V = 3 \int_{-2}^2 (2 - |x|)^2 dx.$$

4. Use symmetry and integrate.

The integrand is even; on $[0, 2]$, $|x| = x$. Thus

$$V = 6 \int_0^2 (2 - x)^2 dx = 6 \int_0^2 (4 - 4x + x^2) dx.$$

$$V = 6 \left[4x - 2x^2 + \frac{x^3}{3} \right]_0^2.$$

5. Apply the bounds.

$$V = 6 \left[\left(8 - 8 + \frac{8}{3} \right) - 0 \right] = 6 \cdot \frac{8}{3} = \boxed{16}.$$

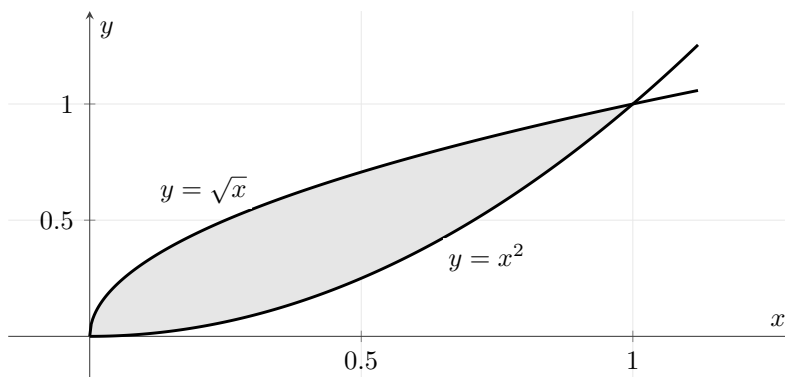
All volumes are in cubic units.

Question 10

The base of a solid is the region bounded by

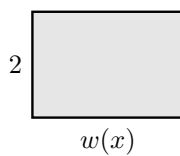
$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

between their points of intersection. Cross-sections parallel to the y -axis are rectangles whose width is equal to the distance between the curves and whose height is constant and equal to 2. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 10 – Solution

1. Interpret the slice direction and find the bounds.

The base segments are parallel to the y -axis, so they are vertical; integrate with respect to x . Solving $\sqrt{x} = x^2$ gives $x = x^4$, or $x(x^3 - 1) = 0$. The real intersections occur at $x = 0, 1$.

2. Identify width and height.

For $0 \leq x \leq 1$, $\sqrt{x} \geq x^2$. Thus each rectangle has

$$w(x) = \sqrt{x} - x^2, \quad h = 2.$$

The height of 2 extends out of the base plane.

3. Find the area and set up the integral.

$$A(x) = 2(\sqrt{x} - x^2), \quad V = \int_0^1 A(x) dx = 2 \int_0^1 (x^{1/2} - x^2) dx.$$

4. Integrate using the power rule.

$$\int x^{1/2} dx = \frac{2}{3}x^{3/2}, \quad \int x^2 dx = \frac{x^3}{3}.$$
$$V = 2 \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3} \right]_0^1.$$

5. Apply the bounds and simplify.

$$V = 2 \left[\left(\frac{2}{3} - \frac{1}{3} \right) - 0 \right] = \boxed{\frac{2}{3}}.$$

All volumes are in cubic units.