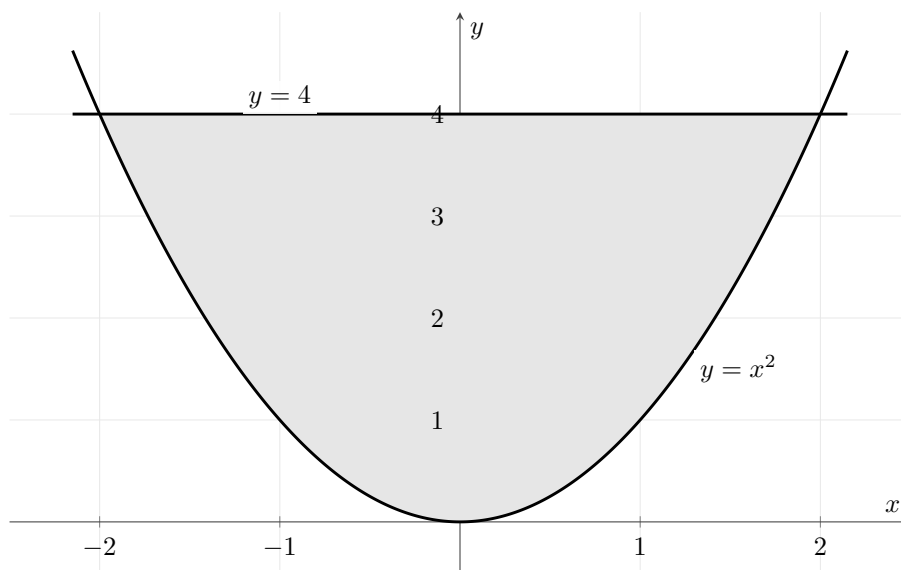


Question 1

Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 \quad \text{and} \quad y = 4$$

about the y -axis.



Rotate the shaded region about the y -axis.

Question 1 – Solution

1. Find the intersections and choose slices.

For shells about the y -axis, use vertical slices parallel to the axis. The curves meet where $x^2 = 4$, so $x = \pm 2$.

2. Choose an interval that counts each shell once.

The half-region $0 \leq x \leq 2$ generates the entire solid. The negative half generates the same shells, so do not double the volume or integrate over both halves.

3. Identify radius, height, and the shell integral.

The radius is the distance to the y -axis; the height is top minus bottom:

$$r(x) = x, \quad h(x) = 4 - x^2.$$
$$V = 2\pi \int_0^2 r(x)h(x) dx = 2\pi \int_0^2 x(4 - x^2) dx.$$

4. Expand and integrate.

$$x(4 - x^2) = 4x - x^3, \quad \int (4x - x^3) dx = 2x^2 - \frac{x^4}{4}.$$

5. Apply the bounds and simplify.

$$\begin{aligned} V &= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 \\ &= 2\pi \left[\left(8 - \frac{16}{4} \right) - 0 \right] = 2\pi(4) = \boxed{8\pi}. \end{aligned}$$

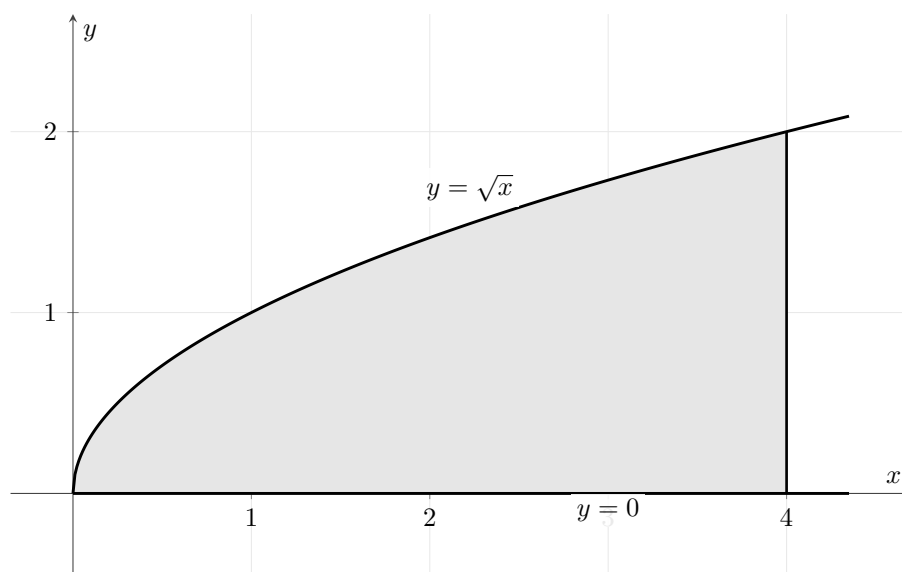
All volumes are in cubic units.

Question 2

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 4$, about the x -axis.



Rotate the shaded region about the x -axis.

Question 2 – Solution

1. Choose horizontal slices and y -bounds.

Shell slices run parallel to the x -axis. Rewrite $y = \sqrt{x}$ as $x = y^2$. The region extends from $y = 0$ to $y = \sqrt{4} = 2$.

2. Identify the shell radius and height.

The radius is the distance to $y = 0$. For a horizontal slice, shell height is its horizontal length, right minus left:

$$r(y) = y, \quad h(y) = 4 - y^2.$$

3. Set up the shell integral.

A thin shell has volume approximately $(2\pi r)(h) dy$. Thus

$$V = 2\pi \int_0^2 y(4 - y^2) dy = 2\pi \int_0^2 (4y - y^3) dy.$$

4. Integrate with respect to y .

$$\int 4y dy = 2y^2, \quad \int y^3 dy = \frac{y^4}{4}.$$
$$V = 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2.$$

5. Evaluate both endpoints.

$$V = 2\pi \left[\left(2(2)^2 - \frac{2^4}{4} \right) - 0 \right] = 2\pi(8 - 4) = \boxed{8\pi}.$$

All volumes are in cubic units.

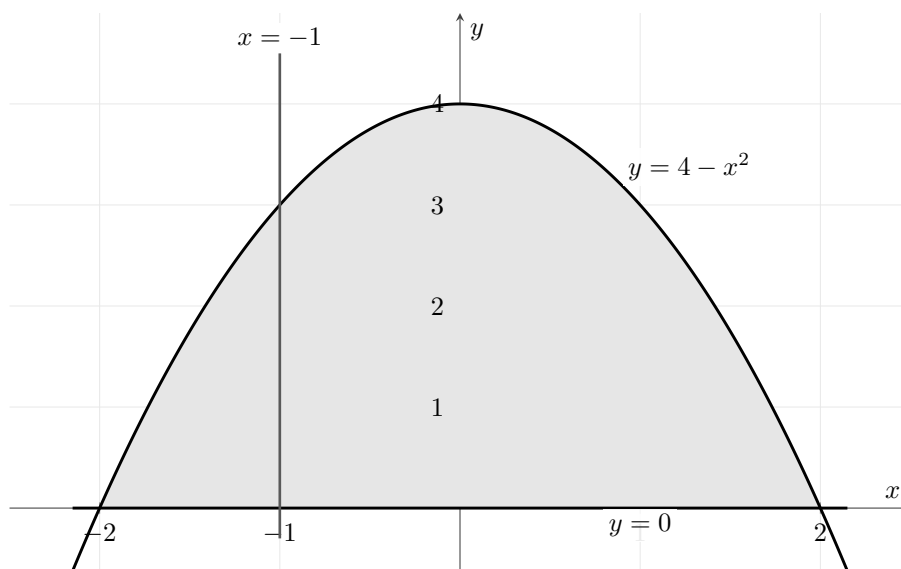
Question 3

Find the volume of the solid obtained by rotating the region bounded by

$$y = 4 - x^2 \quad \text{and} \quad y = 0,$$

about the vertical line

$$x = -1.$$



Rotate the shaded region about the line $x = -1$.

Question 3 – Solution**1. Locate the region and rotation axis.**

The parabola meets $y = 0$ at $x = \pm 2$. The axis $x = -1$ passes through the region. Vertical strips on opposite sides can rotate into the same shell.

2. Count each radius once.

Let r be distance from $x = -1$. The right strip is at $x = -1 + r$, with $0 \leq r \leq 3$. Its height is $4 - (r - 1)^2$. For $0 \leq r \leq 1$, a left strip also exists at $x = -1 - r$, with height $4 - (r + 1)^2$.

$$[4 - (r - 1)^2] - [4 - (r + 1)^2] = 4r \geq 0.$$

Both strips start at $y = 0$, so the taller right strip includes the entire left strip after rotation. Use only the right strip.

3. Set up the shell integral.

$$\begin{aligned} h(r) &= 4 - (r - 1)^2 = 3 + 2r - r^2. \\ V &= 2\pi \int_0^3 r(3 + 2r - r^2) dr = 2\pi \int_0^3 (3r + 2r^2 - r^3) dr. \end{aligned}$$

4. Integrate each power.

$$V = 2\pi \left[\frac{3}{2}r^2 + \frac{2}{3}r^3 - \frac{1}{4}r^4 \right]_0^3.$$

5. Apply the bounds.

$$V = 2\pi \left(\frac{27}{2} + 18 - \frac{81}{4} \right) = 2\pi \left(\frac{54 + 72 - 81}{4} \right) = \boxed{\frac{45\pi}{2}}.$$

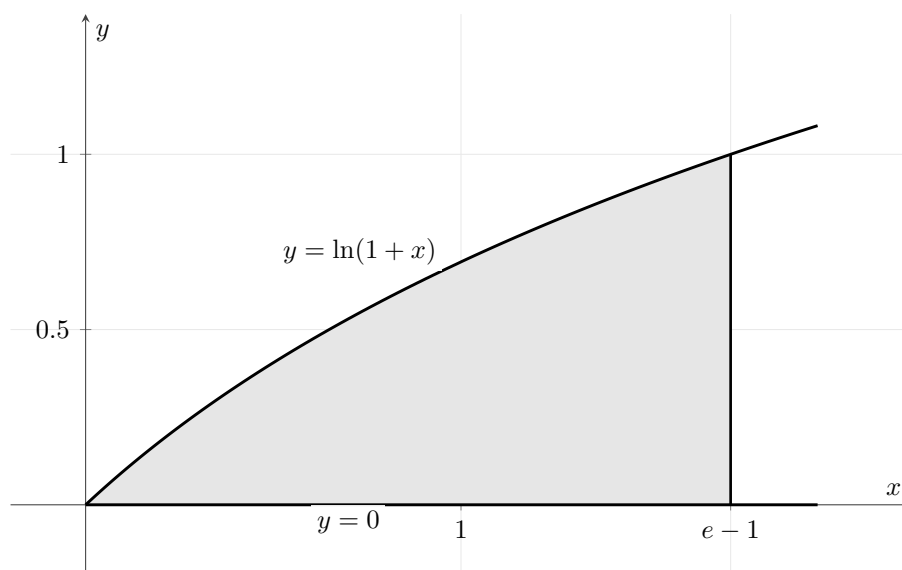
All volumes are in cubic units.

Question 4

Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln(1 + x) \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = e - 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 4 – Solution

1. Identify shell radius and height.

Use vertical slices about the y -axis, with $0 \leq x \leq e - 1$:

$$r(x) = x, \quad h(x) = \ln(1 + x) - 0.$$

2. Set up the volume.

$$V = 2\pi \int_0^{e-1} x \ln(1 + x) dx.$$

3. Apply integration by parts.

Choose $u = \ln(1 + x)$ and $dv = x dx$. Then $du = dx/(1 + x)$ and $v = x^2/2$:

$$\int x \ln(1 + x) dx = \frac{x^2}{2} \ln(1 + x) - \frac{1}{2} \int \frac{x^2}{1 + x} dx.$$

4. Divide and integrate the remaining terms.

Since $x^2 = (x + 1)(x - 1) + 1$,

$$\frac{x^2}{1 + x} = x - 1 + \frac{1}{1 + x}.$$

An antiderivative is therefore

$$F(x) = \frac{x^2}{2} \ln(1 + x) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2} \ln(1 + x).$$

5. Apply the bounds and multiply by 2π .

Using $\ln e = 1$ and $F(0) = 0$,

$$\begin{aligned} F(e - 1) - F(0) &= \frac{(e - 1)^2}{4} + \frac{e - 1}{2} - \frac{1}{2} = \frac{e^2 - 3}{4}, \\ V &= 2\pi \cdot \frac{e^2 - 3}{4} = \boxed{\frac{\pi}{2}(e^2 - 3)}. \end{aligned}$$

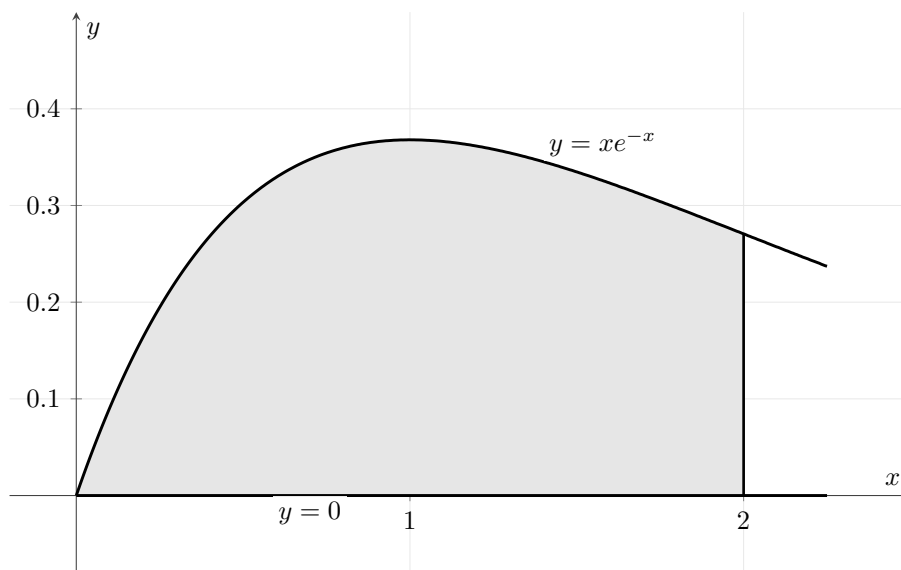
All volumes are in cubic units.

Question 5

Find the volume of the solid obtained by rotating the region bounded by

$$y = xe^{-x} \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 2$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 5 – Solution

1. Identify the shell dimensions.

Use vertical slices for rotation about the y -axis. On $0 \leq x \leq 2$,

$$r(x) = x, \quad h(x) = xe^{-x}.$$

2. Set up the volume integral.

$$V = 2\pi \int_0^2 r(x)h(x) dx = 2\pi \int_0^2 x^2e^{-x} dx.$$

3. Integrate by parts once.

Let $u = x^2$, $dv = e^{-x}dx$, so $du = 2x dx$ and $v = -e^{-x}$:

$$\int x^2e^{-x} dx = -x^2e^{-x} + 2 \int xe^{-x} dx.$$

4. Integrate by parts a second time.

For the remaining integral, choose $u = x$ and $dv = e^{-x}dx$:

$$\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x}.$$

Substitution gives $F(x) = -e^{-x}(x^2 + 2x + 2)$.

5. Evaluate at 2 and 0.

$$\begin{aligned} V &= 2\pi [-e^{-x}(x^2 + 2x + 2)]_0^2 \\ &= 2\pi [-10e^{-2} - (-2)] \\ &= 2\pi(2 - 10e^{-2}) = \boxed{4\pi(1 - 5e^{-2})}. \end{aligned}$$

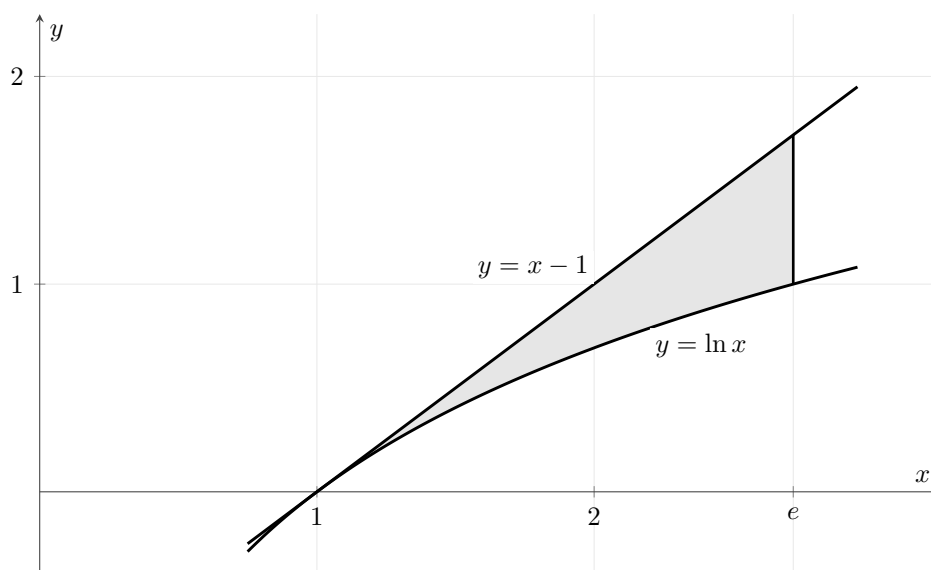
All volumes are in cubic units.

Question 6

Use cylindrical shells to find the volume obtained by revolving the region bounded by

$$y = \ln x, \quad y = x - 1, \quad x = 1, \quad x = e$$

about the y -axis.



Rotate the shaded region about the y -axis.

Question 6 – Solution

1. Determine the upper curve.

Let $g(x) = x - 1 - \ln x$. On $[1, e]$, $g(1) = 0$ and $g'(x) = 1 - 1/x \geq 0$. Hence $x - 1 \geq \ln x$.

2. Identify shell dimensions and set up the volume.

Use vertical slices parallel to the y -axis:

$$r(x) = x, \quad h(x) = x - 1 - \ln x.$$

$$V = 2\pi \int_1^e x(x - 1 - \ln x) dx = 2\pi \int_1^e (x^2 - x - x \ln x) dx.$$

3. Integrate the logarithmic term by parts.

Let $u = \ln x$, $dv = x dx$, so $du = dx/x$ and $v = x^2/2$:

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}.$$

4. Combine the antiderivatives.

$$F(x) = \frac{x^3}{3} - \frac{x^2}{2} - \left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right) = \frac{x^3}{3} - \frac{x^2}{4} - \frac{x^2}{2} \ln x.$$

5. Apply the bounds.

$$F(e) = \frac{e^3}{3} - \frac{3e^2}{4}, \quad F(1) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$V = 2\pi(F(e) - F(1)) = \boxed{\pi \left(\frac{2e^3}{3} - \frac{3e^2}{2} - \frac{1}{6} \right)}.$$

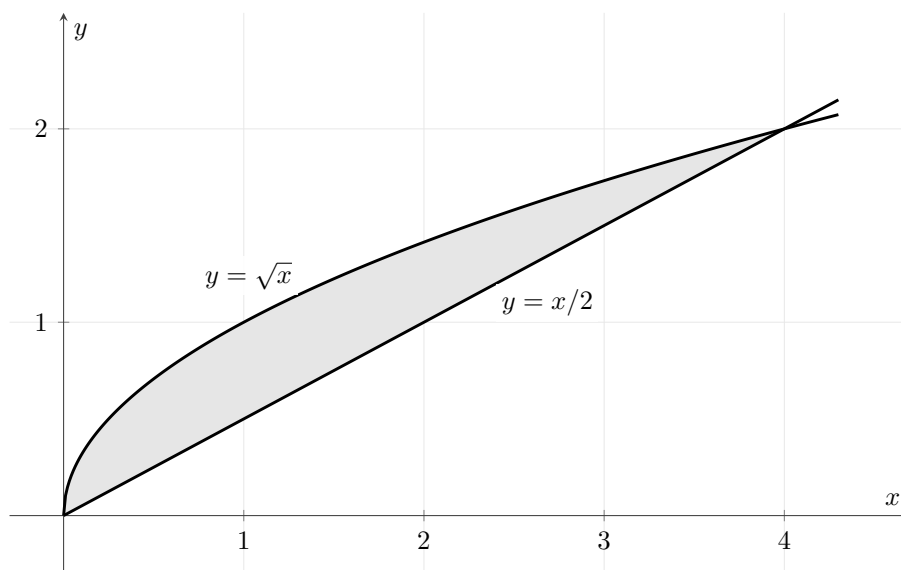
All volumes are in cubic units.

Question 7

Find the volume of the solid obtained by rotating the region bounded by

$$y = \sqrt{x}, \quad y = \frac{x}{2},$$

between their points of intersection, about the y -axis.



Rotate the shaded region about the y -axis.

Question 7 – Solution**1. Find the intersections.**

Both sides of $\sqrt{x} = x/2$ are nonnegative. Squaring and factoring gives

$$x = \frac{x^2}{4} \implies x(x - 4) = 0 \implies x = 0, 4.$$

Both values satisfy the original equation.

2. Identify radius and height.

Use vertical shells about the y -axis. On $[0, 4]$, $\sqrt{x} \geq x/2$:

$$r(x) = x, \quad h(x) = \sqrt{x} - \frac{x}{2}.$$

3. Set up and expand the shell integral.

$$V = 2\pi \int_0^4 x \left(\sqrt{x} - \frac{x}{2} \right) dx = 2\pi \int_0^4 \left(x^{3/2} - \frac{x^2}{2} \right) dx.$$

4. Integrate using the power rule.

$$\int x^{3/2} dx = \frac{2}{5} x^{5/2}, \quad \int \frac{x^2}{2} dx = \frac{x^3}{6}.$$

$$V = 2\pi \left[\frac{2}{5} x^{5/2} - \frac{x^3}{6} \right]_0^4.$$

5. Apply the bounds and simplify.

Since $4^{5/2} = (\sqrt{4})^5 = 32$,

$$V = 2\pi \left(\frac{64}{5} - \frac{64}{6} \right) = 2\pi \left(\frac{192 - 160}{15} \right) = \boxed{\frac{64\pi}{15}}.$$

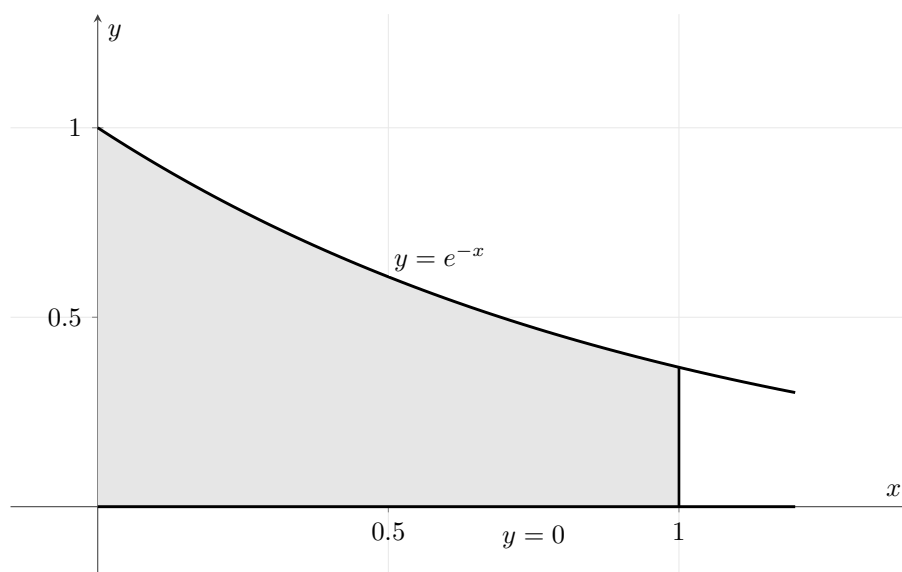
All volumes are in cubic units.

Question 8

Find the volume of the solid obtained by rotating the region bounded by

$$y = e^{-x}, \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 8 – Solution

1. Identify the shell radius and height.

Use vertical slices over $0 \leq x \leq 1$, parallel to the y -axis:

$$r(x) = x, \quad h(x) = e^{-x} - 0 = e^{-x}.$$

2. Set up the volume integral.

Circumference times height times thickness gives

$$V = 2\pi \int_0^1 r(x)h(x) dx = 2\pi \int_0^1 xe^{-x} dx.$$

3. Apply integration by parts.

Let $u = x$ and $dv = e^{-x}dx$, so $du = dx$ and $v = -e^{-x}$:

$$\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x}.$$

4. Evaluate the definite integral.

Factor the antiderivative and substitute the bounds:

$$\int_0^1 xe^{-x} dx = [-(x+1)e^{-x}]_0^1 = -\frac{2}{e} - (-1) = 1 - \frac{2}{e}.$$

5. Multiply by the circumference factor.

$$V = \boxed{2\pi \left(1 - \frac{2}{e}\right)}.$$

Since $e > 2$, the expression is positive, as a volume must be.

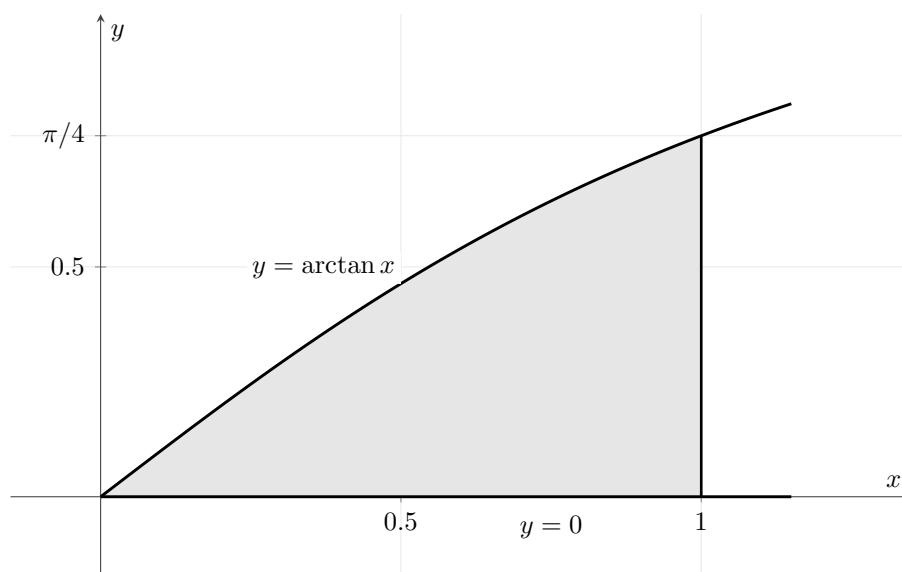
All volumes are in cubic units.

Question 9

Find the volume of the solid obtained by rotating the region bounded by

$$y = \arctan x \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 9 – Solution**1. Identify shell radius and height.**

Use vertical slices about the y -axis on $[0, 1]$:

$$r(x) = x, \quad h(x) = \arctan x.$$

Angles are measured in radians.

2. Set up the volume integral.

$$V = 2\pi \int_0^1 x \arctan x \, dx.$$

3. Apply integration by parts.

Let $u = \arctan x$, $dv = x \, dx$, so $du = dx/(1 + x^2)$ and $v = x^2/2$:

$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1 + x^2} \, dx.$$

4. Simplify the remaining rational function.

$$\frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2}.$$

Since $\int (1 + x^2)^{-1} dx = \arctan x$, an antiderivative is

$$F(x) = \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{1}{2} \arctan x.$$

5. Apply the bounds and simplify.

Using $\arctan 1 = \pi/4$ and $\arctan 0 = 0$,

$$F(1) - F(0) = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.$$

$$V = 2\pi \left(\frac{\pi}{4} - \frac{1}{2} \right) = \boxed{\pi \left(\frac{\pi}{2} - 1 \right)}.$$

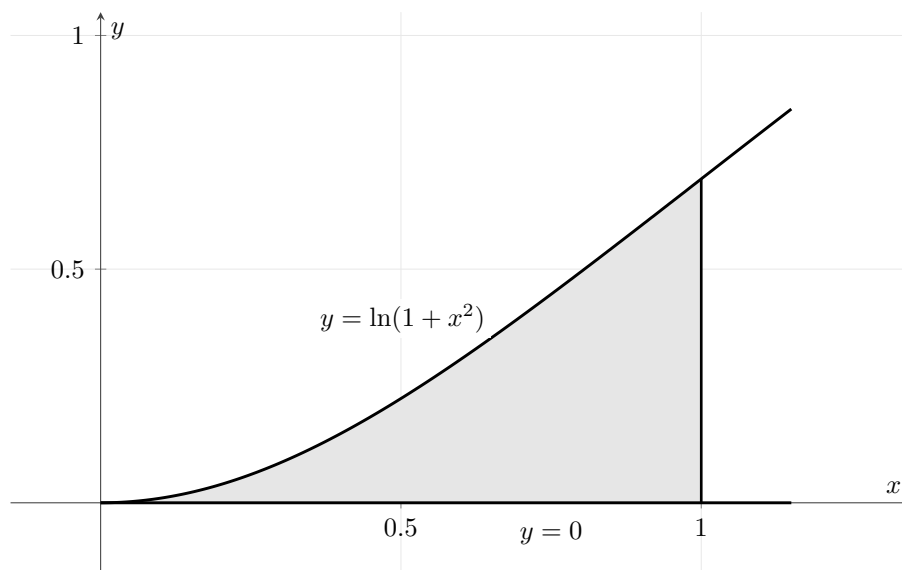
All volumes are in cubic units.

Question 10

Find the volume of the solid obtained by rotating the region bounded by

$$y = \ln(1 + x^2) \quad \text{and} \quad y = 0,$$

from $x = 0$ to $x = 1$, about the y -axis.



Rotate the shaded region about the y -axis.

Question 10 – Solution**1. Identify shell dimensions and set up the integral.**

Use vertical shells about the y -axis, with $0 \leq x \leq 1$:

$$r(x) = x, \quad h(x) = \ln(1 + x^2).$$

$$V = 2\pi \int_0^1 x \ln(1 + x^2) dx.$$

2. Substitute and change the bounds.

Let $u = 1 + x^2$. Then $du = 2x dx$, so $x dx = du/2$. The bounds change as follows:

$$x = 0 \Rightarrow u = 1, \quad x = 1 \Rightarrow u = 2.$$

3. Rewrite the volume in terms of u .

$$V = 2\pi \cdot \frac{1}{2} \int_1^2 \ln u du = \pi \int_1^2 \ln u du.$$

4. Integrate the logarithm by parts.

Choose $a = \ln u$ and $db = du$, giving $da = du/u$ and $b = u$:

$$\int \ln u du = u \ln u - \int u \frac{1}{u} du = u \ln u - u.$$

5. Evaluate the endpoints.

$$\begin{aligned} V &= \pi [u \ln u - u]_1^2 \\ &= \pi [(2 \ln 2 - 2) - (\ln 1 - 1)] \\ &= \pi(2 \ln 2 - 1) = \boxed{2\pi \left(\ln 2 - \frac{1}{2} \right)}. \end{aligned}$$

All volumes are in cubic units.