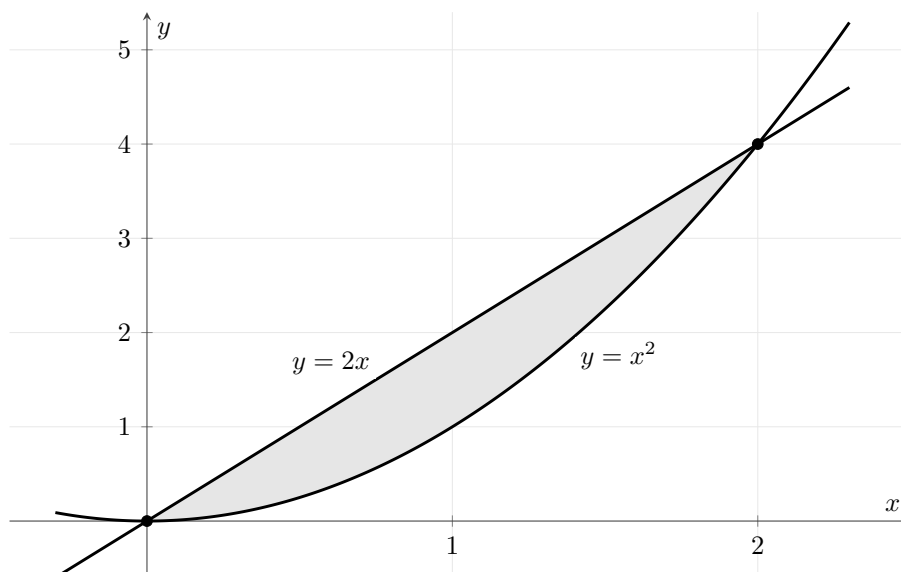


Question 1

Find the exact area of the region enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = 2x,$$

between their points of intersection.



The shaded region is the area to be found.

Question 1 – Solution

1. Find the intersections.

$$x^2 = 2x \implies x(x - 2) = 0 \implies x = 0, 2.$$

Factoring gives $x = 0$ or $x - 2 = 0$. Substituting into $y = 2x$ gives $y = 0$ or $y = 4$. The intersection points are $(0, 0)$ and $(2, 4)$.

2. Determine which curve is above the other.

On $0 \leq x \leq 2$, $2x - x^2 = x(2 - x) \geq 0$, so the line is above the parabola.

3. Set up upper minus lower.

$$A = \int_0^2 (\text{upper} - \text{lower}) dx = \int_0^2 (2x - x^2) dx.$$

4. Find an antiderivative.

By the power rule, $\int 2x dx = x^2$ and $\int x^2 dx = x^3/3$.

5. Apply the bounds and simplify.

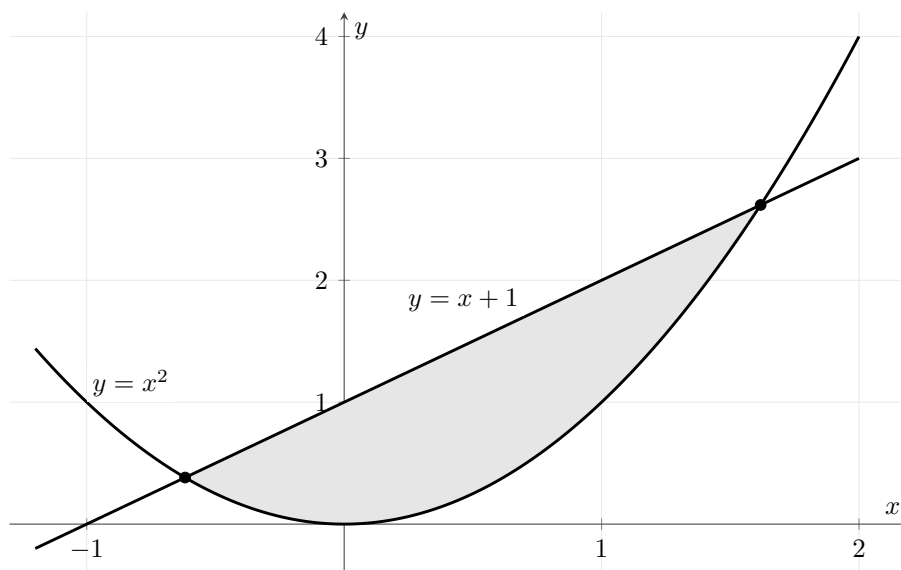
$$\begin{aligned} A &= \left[x^2 - \frac{x^3}{3} \right]_0^2 \\ &= \left(4 - \frac{8}{3} \right) - 0 = \boxed{\frac{4}{3}}. \end{aligned}$$

Question 2

Find the exact area of the region enclosed by the curves

$$y = x + 1 \quad \text{and} \quad y = x^2,$$

between their points of intersection.



The shaded region is the area to be found.

Question 2 – Solution

1. Find the intersections.

$$x^2 = x + 1 \implies x^2 - x - 1 = 0.$$

The quadratic formula gives $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$. Write the two roots as

$$a = \frac{1 - \sqrt{5}}{2}, \quad b = \frac{1 + \sqrt{5}}{2}.$$

2. Determine which curve is above the other.

Test $x = 0$: the line has height 1 and the parabola has height 0. Between the roots, $(x + 1) - x^2 \geq 0$, so the line is above the parabola.

3. Set up upper minus lower.

$$A = \int_a^b (-x^2 + x + 1) dx.$$

4. Complete the square to simplify the bounds.

Complete the square:

$$-x^2 + x + 1 = \frac{5}{4} - \left(x - \frac{1}{2}\right)^2.$$

Let $u = x - \frac{1}{2}$. The new bounds are $-\sqrt{5}/2$ and $\sqrt{5}/2$. Here $du = dx$. The integrand is even, so the two halves have equal area.

5. Integrate and simplify.

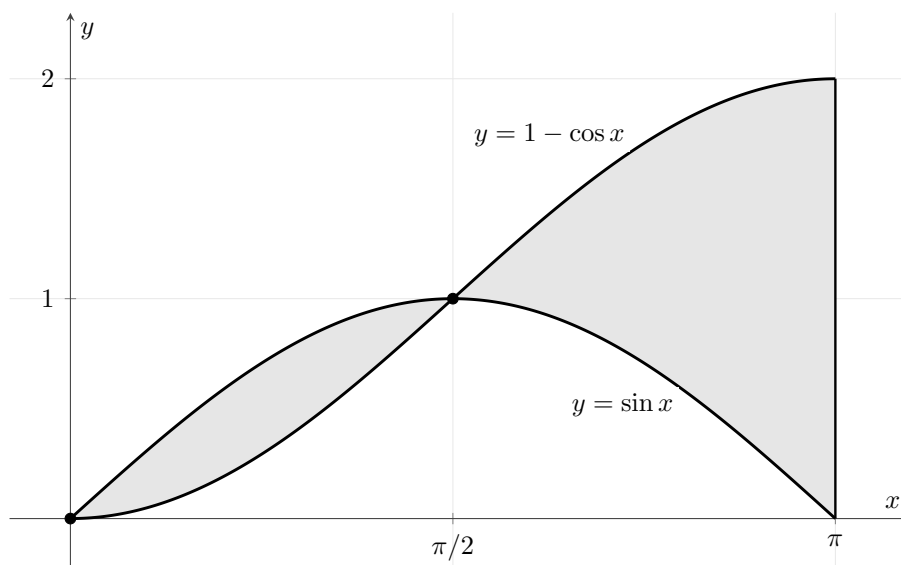
$$\begin{aligned} A &= 2 \int_0^{\sqrt{5}/2} \left(\frac{5}{4} - u^2 \right) du \\ &= 2 \left[\frac{5}{4}u - \frac{u^3}{3} \right]_0^{\sqrt{5}/2} \\ &= 2 \left(\frac{5\sqrt{5}}{8} - \frac{5\sqrt{5}}{24} \right) = \boxed{\frac{5\sqrt{5}}{6}}. \end{aligned}$$

Question 3

Find the exact area of the region between the curves

$$y = \sin x \quad \text{and} \quad y = 1 - \cos x,$$

between $x = 0$ and $x = \pi$.



The shaded region is the area to be found.

Question 3 – Solution

1. Find where the curves cross on $[0, \pi]$.

Using half-angle identities,

$$\begin{aligned}\sin x = 1 - \cos x &\iff 2 \sin(x/2) (\cos(x/2) - \sin(x/2)) = 0 \\ &\iff x = 0 \quad \text{or} \quad x = \frac{\pi}{2}.\end{aligned}$$

2. Determine the upper curve on each interval.

On $(0, \pi/2)$, $\sin x > 1 - \cos x$; on $(\pi/2, \pi)$, the order reverses. Test $x = \pi/4$: $\sin x = \sqrt{2}/2 > 1 - \sqrt{2}/2$. At $x = 3\pi/4$, $1 - \cos x = 1 + \sqrt{2}/2 > \sqrt{2}/2$. The line $x = \pi$ closes the region on the right.

3. Set up the sum of the two areas.

$$A = \int_0^{\pi/2} (\sin x - 1 + \cos x) dx + \int_{\pi/2}^{\pi} (1 - \cos x - \sin x) dx.$$

4. Integrate each piece and apply its bounds.

$$A_1 = [-\cos x - x + \sin x]_0^{\pi/2} = 2 - \frac{\pi}{2},$$

$$A_2 = [x - \sin x + \cos x]_{\pi/2}^{\pi} = (\pi - 1) - \left(\frac{\pi}{2} - 1\right) = \frac{\pi}{2}.$$

5. Add the positive areas.

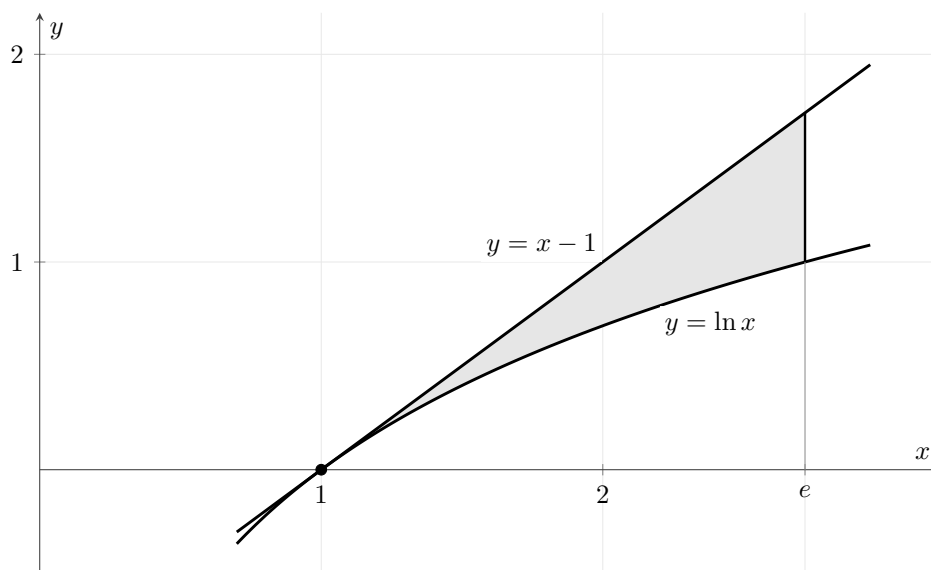
$$A = A_1 + A_2 = \left(2 - \frac{\pi}{2}\right) + \frac{\pi}{2} = \boxed{2}.$$

Question 4

Find the exact area of the region between the curves

$$y = \ln x \quad \text{and} \quad y = x - 1,$$

between $x = 1$ and $x = e$.



The shaded region is the area to be found.

Question 4 – Solution

1. Identify the upper curve on $[1, e]$.

Let $h(x) = x - 1 - \ln x$. Then $h(1) = 0$ and

$$h'(x) = 1 - \frac{1}{x} \geq 0 \quad (1 \leq x \leq e).$$

Thus $x - 1 \geq \ln x$ throughout the interval.

2. Set up the area.

$$A = \int_1^e (x - 1 - \ln x) dx.$$

3. Find the logarithm antiderivative.

Integration by parts, with $u = \ln x$ and $dv = dx$, gives

$$\int \ln x dx = x \ln x - \int 1 dx = x \ln x - x.$$

Here $du = dx/x$ and $v = x$. Combine the terms: $x^2/2 - x - (x \ln x - x) = x^2/2 - x \ln x$.

4. Apply the upper and lower bounds.

$$\begin{aligned} A &= \left[\frac{x^2}{2} - x - (x \ln x - x) \right]_1^e \\ &= \left[\frac{x^2}{2} - x \ln x \right]_1^e \\ &= \left(\frac{e^2}{2} - e \right) - \frac{1}{2} = \boxed{\frac{e^2}{2} - e - \frac{1}{2}}. \end{aligned}$$

5. Check the result.

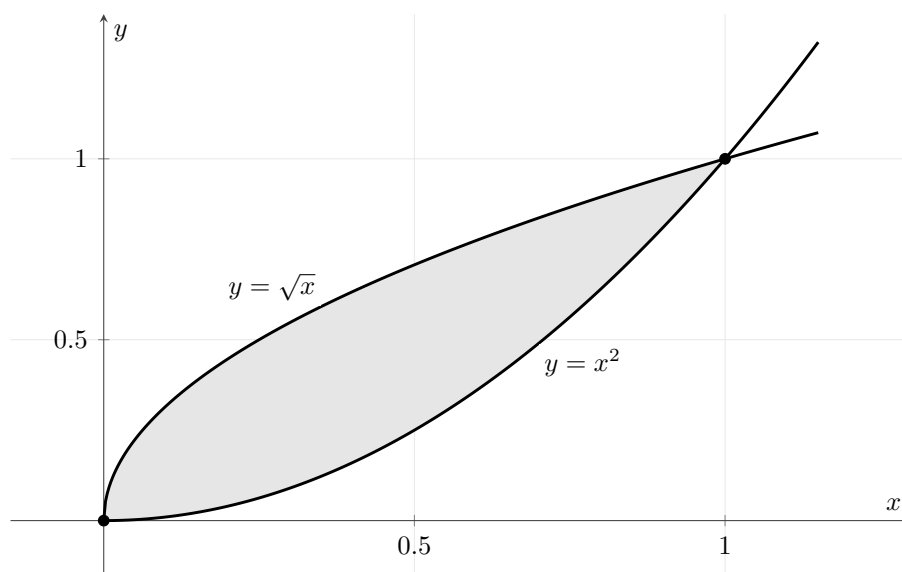
Since $\ln e = 1$ and $\ln 1 = 0$, the endpoint terms above give an area of approximately 0.4762 square units, which is positive.

Question 5

Find the exact area of the region enclosed by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

between their points of intersection.



The shaded region is the area to be found.

Question 5 – Solution

1. Find the intersections.

The square root requires $x \geq 0$. Both sides are nonnegative, so squaring gives

$$\sqrt{x} = x^2 \implies x = x^4 \implies x(x^3 - 1) = 0.$$

The factor $x^3 - 1 = (x - 1)(x^2 + x + 1)$ has only one real zero, $x = 1$. The real solutions $x = 0$ and $x = 1$ both satisfy the original equation. The intersection points are $(0, 0)$ and $(1, 1)$.

2. Determine which curve is above the other.

For $0 < x < 1$, $x^{1/2} > x^2$. For example, at $x = 1/4$, $\sqrt{x} = 1/2$ while $x^2 = 1/16$.

3. Set up upper minus lower.

$$A = \int_0^1 (\sqrt{x} - x^2) dx.$$

4. Integrate each power.

Use $\int x^n dx = x^{n+1}/(n+1)$ for $n \neq -1$:

$$\int x^{1/2} dx = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}, \quad \int x^2 dx = \frac{x^3}{3}.$$

5. Apply the bounds and subtract.

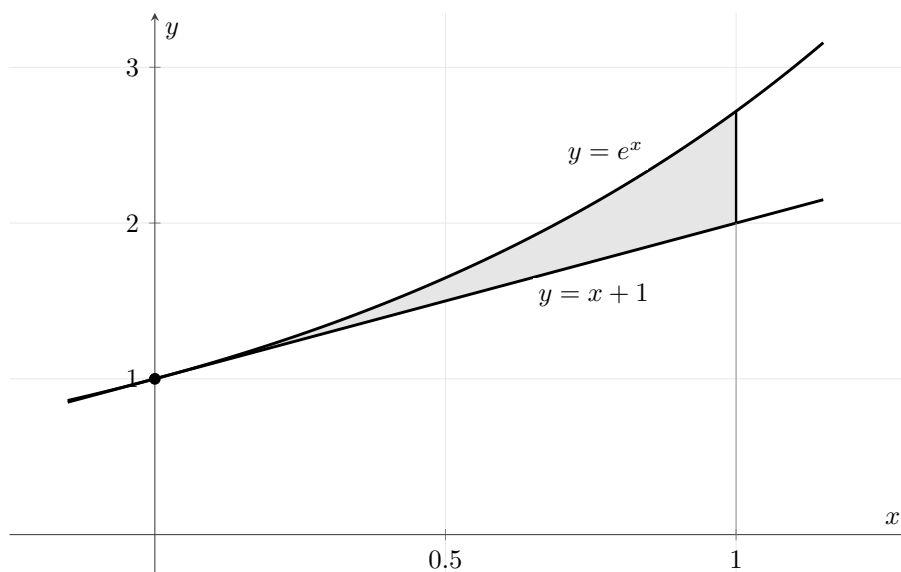
$$\begin{aligned} A &= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3} \right]_0^1 \\ &= \left(\frac{2}{3} - \frac{1}{3} \right) - 0 = \boxed{\frac{1}{3}}. \end{aligned}$$

Question 6

Find the exact area of the region bounded by the curves

$$y = e^x \quad \text{and} \quad y = x + 1,$$

and the vertical lines $x = 0$ and $x = 1$.



The shaded region is the area to be found.

Question 6 – Solution

1. Identify the upper curve on $[0, 1]$.

Let $h(x) = e^x - x - 1$. Then $h(0) = 0$ and

$$h'(x) = e^x - 1 \geq 0 \quad (0 \leq x \leq 1).$$

Thus $e^x \geq x + 1$, with equality only at $x = 0$.

2. Use the given vertical boundaries.

$$A = \int_0^1 (e^x - (x + 1)) \, dx.$$

3. Integrate term by term.

$$\int e^x \, dx = e^x, \quad \int x \, dx = \frac{x^2}{2}, \quad \int 1 \, dx = x.$$

4. Evaluate at $x = 1$ and subtract the value at $x = 0$.

$$\begin{aligned} A &= \left[e^x - \frac{x^2}{2} - x \right]_0^1 \\ &= \left(e - \frac{1}{2} - 1 \right) - (1 - 0 - 0) \\ &= \boxed{e - \frac{5}{2}}. \end{aligned}$$

5. Check the sign and units.

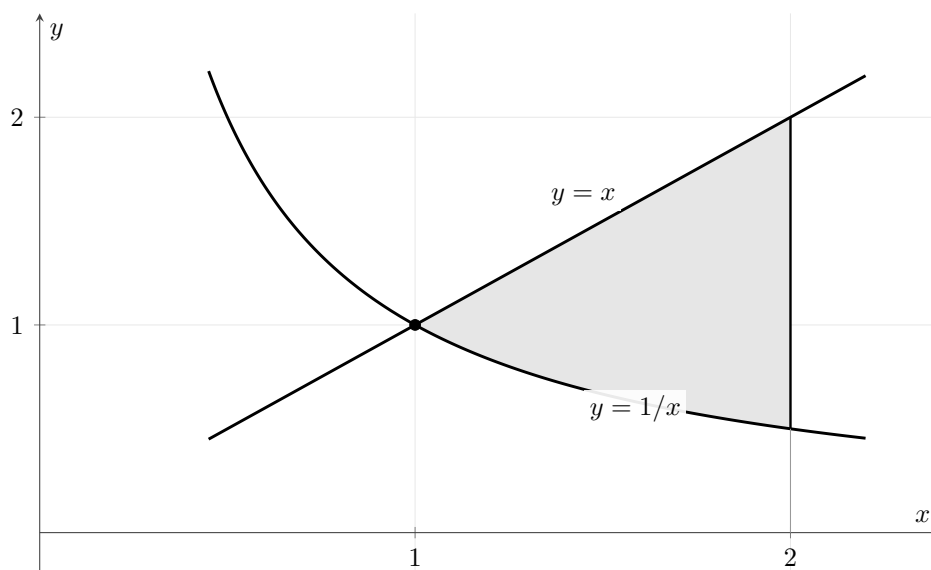
The area is approximately $2.71828 - 2.5 = 0.21828$ square units. It is positive because e^x lies above the line on $(0, 1]$.

Question 7

Find the exact area of the region between the curves

$$y = \frac{1}{x} \quad \text{and} \quad y = x,$$

between $x = 1$ and $x = 2$.



The shaded region is the area to be found.

Question 7 – Solution

1. Identify the upper curve on $[1, 2]$.

For $x \geq 1$,

$$x - \frac{1}{x} = \frac{x^2 - 1}{x} \geq 0.$$

Thus the line $y = x$ lies above $y = 1/x$. They meet at $(1, 1)$, and the boundary $x = 2$ closes the region.

2. Set up the area.

$$A = \int_1^2 \left(x - \frac{1}{x} \right) dx.$$

3. Integrate each term.

Since $x > 0$ on this interval, an antiderivative of $1/x$ is $\ln x$.

$$\int x \, dx = \frac{x^2}{2}, \quad \int \frac{1}{x} \, dx = \ln x.$$

4. Substitute the bounds and simplify.

$$\begin{aligned} A &= \left[\frac{x^2}{2} - \ln x \right]_1^2 \\ &= (2 - \ln 2) - \left(\frac{1}{2} - \ln 1 \right) \\ &= \boxed{\frac{3}{2} - \ln 2}. \end{aligned}$$

5. Check the result.

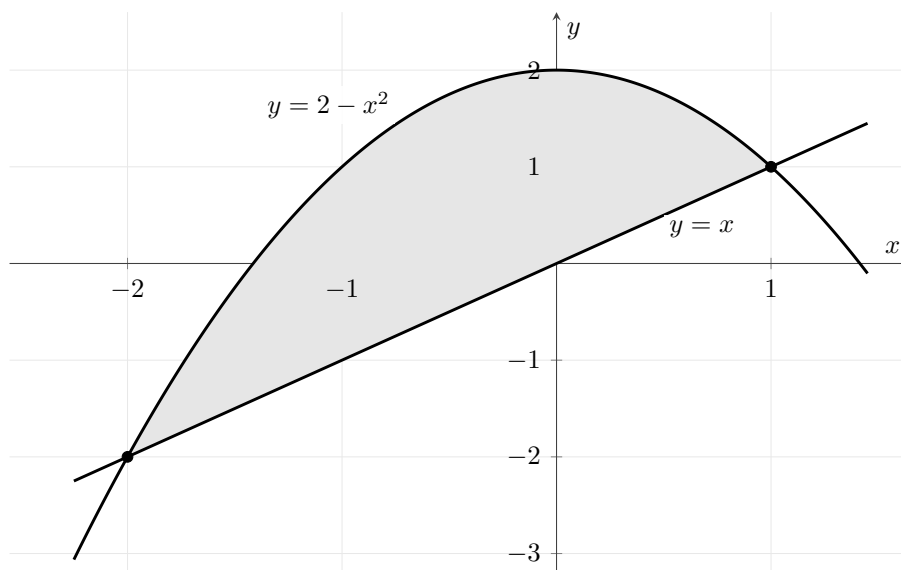
Using $\ln 2 \approx 0.69315$, the area is $1.5 - 0.69315 \approx 0.80685$ square units. The result is positive, as required.

Question 8

Find the exact area of the region enclosed by the curves

$$y = 2 - x^2 \quad \text{and} \quad y = x,$$

between their points of intersection.



The shaded region is the area to be found.

Question 8 – Solution

1. Find the intersections.

$$2 - x^2 = x \implies x^2 + x - 2 = 0 \implies (x + 2)(x - 1) = 0.$$

The zero-product property gives $x + 2 = 0$ or $x - 1 = 0$. Thus $x = -2$ and $x = 1$, giving points $(-2, -2)$ and $(1, 1)$.

2. Determine which curve is above the other.

On $[-2, 1]$, $(2 - x^2) - x = (1 - x)(x + 2) \geq 0$. The parabola is above the line, even where both curves are below the x -axis.

3. Set up upper minus lower.

$$A = \int_{-2}^1 (2 - x - x^2) dx.$$

4. Find an antiderivative.

$$\int 2 dx = 2x, \quad \int x dx = \frac{x^2}{2}, \quad \int x^2 dx = \frac{x^3}{3}.$$

5. Apply the bounds, taking care with negative signs.

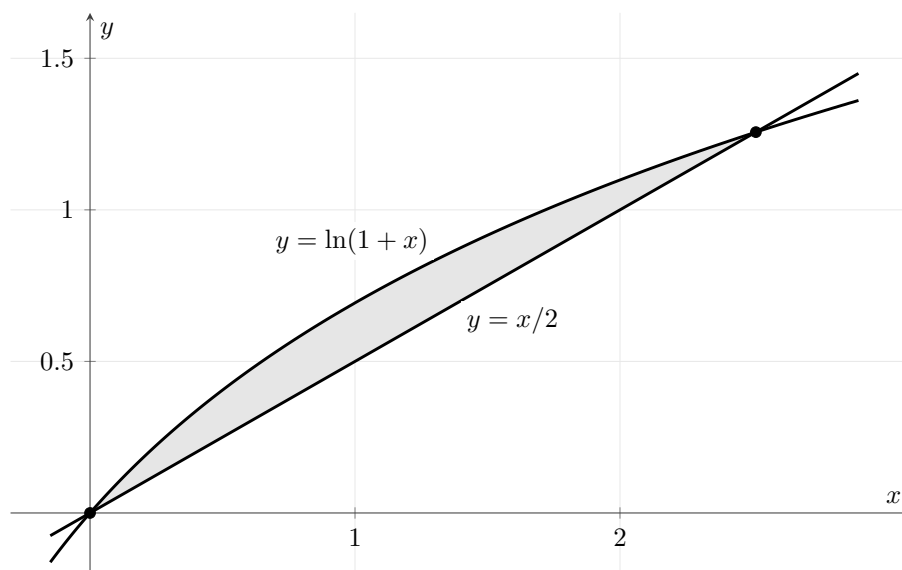
$$\begin{aligned} A &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^1 \\ &= \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - 2 + \frac{8}{3} \right) \\ &= \frac{7}{6} - \left(-\frac{10}{3} \right) = \boxed{\frac{9}{2}}. \end{aligned}$$

Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1 + x) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection.



The shaded region is the area to be found.

Question 9 – Solution

1. Describe the intersections.

Let $h(x) = \ln(1+x) - x/2$, with domain $x > -1$. Then

$$h'(x) = \frac{1-x}{2(1+x)}.$$

So h increases up to $x = 1$ and decreases thereafter. Since $h(0) = 0$, $h(1) > 0$, and $h(x) \rightarrow -\infty$ as $x \rightarrow \infty$, the only roots are 0 and a unique number $a > 1$ satisfying

$$\ln(1+a) = \frac{a}{2}, \quad a \approx 2.512862417.$$

The nonzero root has no elementary closed form; this equation defines a exactly.

2. Set up the area.

On $(0, a)$, $h(x) > 0$, so the logarithm is above the line.

$$A = \int_0^a \left(\ln(1+x) - \frac{x}{2} \right) dx.$$

3. Find an antiderivative.

With $u = 1+x$, integration by parts gives

$$\int \ln(1+x) dx = (1+x) \ln(1+x) - x + C.$$

Also, $\int (x/2) dx = x^2/4$. At $x = 0$, all terms below are zero.

4. Evaluate, then substitute $\ln(1+a) = a/2$.

$$\begin{aligned} A &= \left[(1+x) \ln(1+x) - x - \frac{x^2}{4} \right]_0^a \\ &= (1+a) \frac{a}{2} - a - \frac{a^2}{4} \\ &= \boxed{\frac{a(a-2)}{4}} \approx 0.322188173. \end{aligned}$$

5. Interpret the exact answer.

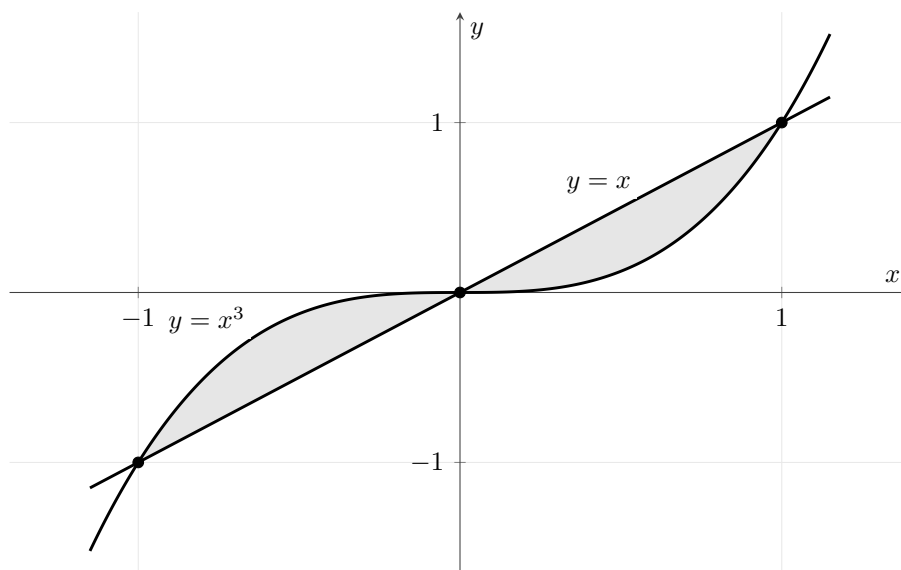
The boxed expression is exact with a defined above; using its numerical approximation gives 0.322188173 square units.

Question 10

Find the exact total area of the regions enclosed by the curves

$$y = x^3 \quad \text{and} \quad y = x,$$

between their points of intersection.



The shaded region is the area to be found.

Question 10 – Solution

1. Find all intersections.

$$x^3 = x \implies x(x-1)(x+1) = 0 \implies x = -1, 0, 1.$$

2. Split the area where the upper curve changes.

On $(-1, 0)$, $x^3 > x$; on $(0, 1)$, $x > x^3$. For example, at $x = -1/2$, $x^3 = -1/8 > -1/2$; at $x = 1/2$, $x = 1/2 > 1/8 = x^3$. Both enclosed regions contribute positive area:

3. Set up upper minus lower on each interval.

$$A = \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx.$$

4. Integrate and evaluate each region.

The power rule gives $\int x^3 dx = x^4/4$ and $\int x dx = x^2/2$.

$$A_1 = \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 = 0 - \left(\frac{1}{4} - \frac{1}{2} \right) = \frac{1}{4},$$

$$A_2 = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

5. Add both areas.

$$A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}.$$