

Question 1

Find the average value of the function

$$f(x) = x\sqrt{1+x^2}$$

on the interval $[0, 2]$.

Question 1 - Solution

The average value of a function f on an interval $[a, b]$ is defined by

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here, $a = 0$ and $b = 2$, so

$$f_{\text{avg}} = \frac{1}{2} \int_0^2 x \sqrt{1+x^2} dx.$$

Evaluate the integral using substitution. Let

$$u = 1 + x^2.$$

Then

$$du = 2x dx \quad \Rightarrow \quad x dx = \frac{1}{2} du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 2$, $u = 5$.

Substitute:

$$\int_0^2 x \sqrt{1+x^2} dx = \frac{1}{2} \int_1^5 u^{1/2} du.$$

Integrate:

$$\frac{1}{2} \int u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} = \frac{1}{3} u^{3/2}.$$

Apply the limits:

$$\left. \frac{1}{3} u^{3/2} \right|_1^5 = \frac{1}{3} (5^{3/2} - 1).$$

Now divide by the interval length:

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{1}{3} (5^{3/2} - 1) = \frac{1}{6} (5^{3/2} - 1).$$

$$\boxed{\frac{1}{6} (5^{3/2} - 1)}$$

Question 2

Find the average value of the function

$$f(x) = \ln(1 + x^2)$$

on the interval $[-1, 1]$.

Question 2 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here $a = -1$ and $b = 1$, so

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \ln(1+x^2) dx.$$

Notice that $\ln(1+x^2)$ is an even function, since

$$\ln(1+(-x)^2) = \ln(1+x^2).$$

Therefore,

$$\int_{-1}^1 \ln(1+x^2) dx = 2 \int_0^1 \ln(1+x^2) dx.$$

Thus,

$$f_{\text{avg}} = \int_0^1 \ln(1+x^2) dx.$$

Evaluate the integral using integration by parts. Let

$$u = \ln(1+x^2), \quad dv = dx.$$

Then

$$du = \frac{2x}{1+x^2} dx, \quad v = x.$$

Apply integration by parts:

$$\int \ln(1+x^2) dx = x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx.$$

Rewrite the integrand:

$$\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2}.$$

Thus,

$$\int \ln(1+x^2) dx = x \ln(1+x^2) - \int \left(2 - \frac{2}{1+x^2} \right) dx.$$

Integrate:

$$x \ln(1+x^2) - 2x + 2 \arctan x.$$

Evaluate from 0 to 1:

$$\left[x \ln(1+x^2) - 2x + 2 \arctan x \right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$

Therefore,

$$\boxed{\ln 2 - 2 + \frac{\pi}{2}}$$

Question 3

Find the average value of the function

$$f(x) = x^2 \cos x$$

on the interval $[0, \pi]$.

Question 3 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{\pi} \int_0^\pi x^2 \cos x dx.$$

Evaluate the integral using integration by parts.

Let

$$u = x^2, \quad dv = \cos x dx.$$

Then

$$du = 2x dx, \quad v = \sin x.$$

Apply integration by parts:

$$\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = 2x, \quad dv = \sin x dx.$$

Then

$$du = 2 dx, \quad v = -\cos x.$$

Thus,

$$\int 2x \sin x dx = -2x \cos x + 2 \int \cos x dx = -2x \cos x + 2 \sin x.$$

Substitute back:

$$\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x.$$

Evaluate from 0 to π :

$$\left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^\pi.$$

At $x = \pi$:

$$\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi = -2\pi.$$

At $x = 0$:

$$0 + 0 - 0 = 0.$$

Thus,

$$\int_0^\pi x^2 \cos x dx = -2\pi.$$

Divide by the interval length:

$$f_{\text{avg}} = \frac{1}{\pi}(-2\pi) = -2.$$

$$\boxed{-2}$$

Question 4

Find the average value of the function

$$f(x) = \sqrt{x}(1+x)$$

on the interval $[0, 1]$.

Question 4 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \int_0^1 \sqrt{x}(1+x) dx.$$

First expand the integrand:

$$\sqrt{x}(1+x) = x^{1/2} + x^{3/2}.$$

Integrate term by term:

$$\int x^{1/2} dx = \frac{2}{3}x^{3/2}, \quad \int x^{3/2} dx = \frac{2}{5}x^{5/2}.$$

Thus,

$$\int_0^1 \sqrt{x}(1+x) dx = \left[\frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} \right]_0^1.$$

Evaluate:

$$\frac{2}{3} + \frac{2}{5} = \frac{10+6}{15} = \frac{16}{15}.$$

$$\boxed{\frac{16}{15}}$$

Question 5

Find the average value of the function

$$f(x) = \frac{x}{1+x^2}$$

on the interval $[0, 1]$.

Question 5 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here $a = 0$ and $b = 1$, so

$$f_{\text{avg}} = \int_0^1 \frac{x}{1+x^2} dx.$$

Evaluate the integral using substitution.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x dx \quad \Rightarrow \quad x dx = \frac{1}{2} du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute:

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{1}{u} du.$$

Integrate:

$$\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln u.$$

Apply the limits:

$$\frac{1}{2} \ln u \Big|_1^2 = \frac{1}{2} \ln 2.$$

$$\boxed{\frac{1}{2} \ln 2}$$

Question 6

Find the average value of the function

$$f(x) = \sin^2 x$$

on the interval $[0, \pi]$.

Question 6 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{\pi} \int_0^{\pi} \sin^2 x dx.$$

Use the identity

$$\sin^2 x = \frac{1 - \cos(2x)}{2}.$$

Then

$$\int_0^{\pi} \sin^2 x dx = \frac{1}{2} \int_0^{\pi} (1 - \cos 2x) dx.$$

Integrate:

$$\frac{1}{2} \left[x - \frac{1}{2} \sin(2x) \right]_0^{\pi} = \frac{1}{2} (\pi - 0) = \frac{\pi}{2}.$$

Now divide by the interval length:

$$f_{\text{avg}} = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 7

Find the average value of the function

$$f(x) = \sqrt{1 - x^2}$$

on the interval $[-1, 1]$.

Question 7 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} dx.$$

Method 1: Geometry

The graph of

$$y = \sqrt{1-x^2}$$

is the upper half of a circle of radius 1 centered at the origin.

The area under this curve from $x = -1$ to $x = 1$ is the area of a semicircle:

$$\frac{1}{2} \pi (1)^2 = \frac{\pi}{2}.$$

Therefore,

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

Method 2: Trigonometric Substitution

Evaluate the integral directly.

Let

$$x = \sin \theta, \quad dx = \cos \theta d\theta.$$

Change the limits. When $x = -1$, $\theta = -\frac{\pi}{2}$. When $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute:

$$\int_{-1}^1 \sqrt{1-x^2} dx = \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta.$$

Integrate:

$$\frac{1}{2} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_{-\pi/2}^{\pi/2}.$$

Since $\sin(\pi) = \sin(-\pi) = 0$, this becomes

$$\frac{1}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

$$\boxed{\frac{\pi}{4}}$$

Question 8

Find the average value of the function

$$f(x) = |x|$$

on the interval $[-2, 2]$.

Question 8 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

Here $a = -2$ and $b = 2$, so

$$f_{\text{avg}} = \frac{1}{4} \int_{-2}^2 |x| \, dx.$$

Since $|x|$ is an even function,

$$\int_{-2}^2 |x| \, dx = 2 \int_0^2 x \, dx.$$

Evaluate:

$$2 \left[\frac{x^2}{2} \right]_0^2 = 2 \cdot 2 = 4.$$

Compute the average value:

$$f_{\text{avg}} = \frac{1}{4} \cdot 4 = 1.$$

1

Question 9

Find the average value of the function

$$f(x) = \sqrt{x^2 + 1}$$

on the interval $[-1, 1]$.

Question 9 - Solution

The average value of a function on an interval $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \sqrt{x^2 + 1} dx.$$

Notice that $\sqrt{x^2 + 1}$ is an even function, so

$$\int_{-1}^1 \sqrt{x^2 + 1} dx = 2 \int_0^1 \sqrt{x^2 + 1} dx.$$

Thus,

$$f_{\text{avg}} = \int_0^1 \sqrt{x^2 + 1} dx.$$

Evaluate the integral using trigonometric substitution. Let

$$x = \tan \theta, \quad dx = \sec^2 \theta d\theta.$$

When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{4}$.

Substitute:

$$\int_0^1 \sqrt{x^2 + 1} dx = \int_0^{\pi/4} \sec \theta \cdot \sec^2 \theta d\theta = \int_0^{\pi/4} \sec^3 \theta d\theta.$$

Use the standard result

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).$$

Apply the limits:

$$\frac{1}{2} (\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)) \Big|_0^{\pi/4}.$$

Evaluate:

$$\begin{aligned} \sec\left(\frac{\pi}{4}\right) &= \sqrt{2}, & \tan\left(\frac{\pi}{4}\right) &= 1, \\ \sec(0) &= 1, & \tan(0) &= 0. \end{aligned}$$

Thus,

$$f_{\text{avg}} = \frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right).$$

$$\boxed{\frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}$$

Question 10

Find the average value of the function

$$f(x) = x e^{-x}$$

on the interval $[0, 2]$.

Question 10 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_0^2 x e^{-x} dx.$$

Evaluate the integral using integration by parts.

Let

$$u = x, \quad dv = e^{-x} dx.$$

Then

$$du = dx, \quad v = -e^{-x}.$$

Apply integration by parts:

$$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x}.$$

Thus,

$$\int x e^{-x} dx = -(x+1)e^{-x}.$$

Evaluate from 0 to 2:

$$\left[-(x+1)e^{-x} \right]_0^2 = -3e^{-2} - (-1) = 1 - 3e^{-2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2} (1 - 3e^{-2}).$$

$$\frac{1}{2} (1 - 3e^{-2})$$