

Question 1

Evaluate the definite integral

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} dx.$$

Question 1 - Solution

Use substitution to simplify the integrand.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx \quad \Rightarrow \quad x \, dx = \frac{1}{2} \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} \, dx = \frac{1}{2} \int_1^2 u^{-1/2} \, du.$$

Integrate:

$$\frac{1}{2} \int u^{-1/2} \, du = \frac{1}{2} \cdot 2u^{1/2} = \sqrt{u}.$$

Apply the new limits:

$$\left. \sqrt{u} \right|_1^2 = \sqrt{2} - 1.$$

$\sqrt{2} - 1$

Question 2

Evaluate the definite integral

$$\int_0^{\pi/2} \sin x \sqrt{1 + \cos x} \, dx.$$

Question 2 - Solution

This integral suggests a substitution involving $\cos x$ because its derivative appears in the integrand.

Let

$$u = 1 + \cos x.$$

Then

$$du = -\sin x \, dx \quad \Rightarrow \quad -du = \sin x \, dx.$$

Change the limits of integration. When $x = 0$, $\cos 0 = 1$, so $u = 2$. When $x = \frac{\pi}{2}$, $\cos(\frac{\pi}{2}) = 0$, so $u = 1$.

Substitute into the integral:

$$\int_0^{\pi/2} \sin x \sqrt{1 + \cos x} \, dx = - \int_2^1 \sqrt{u} \, du = \int_1^2 \sqrt{u} \, du.$$

Integrate:

$$\int \sqrt{u} \, du = \frac{2}{3} u^{3/2}.$$

Apply the limits:

$$\left. \frac{2}{3} u^{3/2} \right|_1^2 = \frac{2}{3} (2^{3/2} - 1).$$

$$\boxed{\frac{2}{3} (2^{3/2} - 1)}$$

Question 3

Evaluate the definite integral

$$\int_1^2 \frac{\ln x}{x} dx.$$

Question 3 - Solution

This integral is well suited for substitution because the derivative of $\ln x$ appears in the denominator.

Let

$$u = \ln x.$$

Then

$$du = \frac{1}{x} dx.$$

Change the limits of integration. When $x = 1$, $u = \ln 1 = 0$. When $x = 2$, $u = \ln 2$.

Substitute into the integral:

$$\int_1^2 \frac{\ln x}{x} dx = \int_0^{\ln 2} u du.$$

Integrate:

$$\int u du = \frac{u^2}{2}.$$

Apply the limits:

$$\frac{u^2}{2} \Big|_0^{\ln 2} = \frac{(\ln 2)^2}{2}.$$

$$\boxed{\frac{(\ln 2)^2}{2}}$$

Question 4

Evaluate the definite integral

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}} dx.$$

Question 4 - Solution

This integral suggests a substitution involving the expression inside the square root.

Let

$$u = 1 + x^4.$$

Then

$$du = 4x^3 dx \quad \Rightarrow \quad x^3 dx = \frac{1}{4} du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x^3}{\sqrt{1+x^4}} dx = \frac{1}{4} \int_1^2 u^{-1/2} du.$$

Integrate:

$$\frac{1}{4} \int u^{-1/2} du = \frac{1}{4} \cdot 2u^{1/2} = \frac{1}{2} \sqrt{u}.$$

Apply the limits:

$$\frac{1}{2} \sqrt{u} \Big|_1^2 = \frac{1}{2} (\sqrt{2} - 1).$$

$$\boxed{\frac{\sqrt{2} - 1}{2}}$$

Question 5

Evaluate the definite integral

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} dx.$$

Question 5 - Solution

This integral is well suited for substitution because the derivative of $\sin x$ appears in the numerator.

Let

$$u = 1 + \sin x.$$

Then

$$du = \cos x \, dx.$$

Change the limits of integration. When $x = 0$, $\sin 0 = 0$, so $u = 1$. When $x = \frac{\pi}{2}$, $\sin(\frac{\pi}{2}) = 1$, so $u = 2$.

Substitute into the integral:

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin x} \, dx = \int_1^2 \frac{1}{u} \, du.$$

Integrate:

$$\int \frac{1}{u} \, du = \ln u.$$

Apply the limits:

$$\ln u \Big|_1^2 = \ln 2.$$

$$\boxed{\ln 2}$$

Question 6

Evaluate the definite integral

$$\int_0^{\sqrt{3}} \frac{x}{1+x^2} dx.$$

Question 6 - Solution

The integrand suggests a substitution involving $1 + x^2$.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx \quad \Rightarrow \quad x \, dx = \frac{1}{2} \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = \sqrt{3}$, $u = 4$.

Substitute into the integral:

$$\int_0^{\sqrt{3}} \frac{x}{1 + x^2} \, dx = \frac{1}{2} \int_1^4 \frac{1}{u} \, du.$$

Integrate:

$$\frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln u.$$

Apply the limits:

$$\frac{1}{2} \ln u \Big|_1^4 = \frac{1}{2} \ln 4 = \ln 2.$$

$$\boxed{\ln 2}$$

Question 7

Evaluate the definite integral

$$\int_1^e \frac{1}{x} \sqrt{1 + \ln x} \, dx.$$

Question 7 - Solution

The integrand suggests a substitution involving $\ln x$.

Let

$$u = 1 + \ln x.$$

Then

$$du = \frac{1}{x} dx.$$

Change the limits of integration. When $x = 1$, $u = 1$. When $x = e$, $u = 2$.

Substitute into the integral:

$$\int_1^e \frac{1}{x} \sqrt{1 + \ln x} dx = \int_1^2 \sqrt{u} du.$$

Integrate:

$$\int \sqrt{u} du = \frac{2}{3} u^{3/2}.$$

Apply the limits:

$$\left. \frac{2}{3} u^{3/2} \right|_1^2 = \frac{2}{3} (2^{3/2} - 1).$$

$$\boxed{\frac{2}{3} (2^{3/2} - 1)}$$

Question 8

Evaluate the definite integral

$$\int_0^1 \frac{2x}{(1+x^2)^2} dx.$$

Question 8 - Solution

The integrand suggests a substitution involving $1 + x^2$.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{2x}{(1+x^2)^2} \, dx = \int_1^2 u^{-2} \, du.$$

Integrate:

$$\int u^{-2} \, du = -u^{-1}.$$

Apply the limits:

$$-u^{-1} \Big|_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$

Question 9

Evaluate the definite integral

$$\int_0^1 \frac{\sqrt{1-x}}{1+x} dx.$$

Question 9 - Solution

This integral becomes simpler if we eliminate the square root by substitution.

Let

$$u = \sqrt{1-x}.$$

Then

$$x = 1 - u^2, \quad dx = -2u \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 0$.

Substitute into the integral:

$$\int_0^1 \frac{\sqrt{1-x}}{1+x} dx = \int_1^0 \frac{u}{1+(1-u^2)} (-2u) du.$$

Simplify:

$$\int_1^0 \frac{-2u^2}{2-u^2} du = \int_0^1 \frac{2u^2}{2-u^2} du.$$

Rewrite the integrand:

$$\frac{2u^2}{2-u^2} = -2 + \frac{4}{2-u^2}.$$

Thus,

$$\int_0^1 \left(-2 + \frac{4}{2-u^2} \right) du.$$

Integrate term by term:

$$\begin{aligned} \int -2 \, du &= -2u, \\ \int \frac{4}{2-u^2} \, du &= 2 \int \frac{1}{1-(u/\sqrt{2})^2} \, du. \end{aligned}$$

Use the standard result

$$\int \frac{1}{1-a^2} da = \operatorname{artanh}(a).$$

Thus,

$$2 \int \frac{1}{1-(u/\sqrt{2})^2} du = 2\sqrt{2} \operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right).$$

Evaluate from 0 to 1:

$$\left. -2u + 2\sqrt{2} \operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right) \right|_0^1.$$

Final answer:

$$-2 + 2\sqrt{2} \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)$$

Question 10

Evaluate the definite integral

$$\int_0^1 \sqrt{1-x^2} \arcsin x \, dx.$$

Question 10 - Solution

This integral is interesting because it mixes an inverse trigonometric function with an algebraic expression, suggesting a trigonometric substitution.

Let

$$x = \sin \theta.$$

Then

$$dx = \cos \theta \, d\theta, \quad \sqrt{1-x^2} = \cos \theta.$$

Change the limits of integration. When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_0^1 \sqrt{1-x^2} \arcsin x \, dx = \int_0^{\pi/2} \theta \cos^2 \theta \, d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_0^{\pi/2} \theta \cos^2 \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \theta \, d\theta + \frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta.$$

The first integral is

$$\frac{1}{2} \int_0^{\pi/2} \theta \, d\theta = \frac{1}{2} \cdot \frac{(\pi/2)^2}{2} = \frac{\pi^2}{16}.$$

For the second integral, use integration by parts. Let

$$u = \theta, \quad dv = \cos(2\theta) \, d\theta.$$

Then

$$du = d\theta, \quad v = \frac{1}{2} \sin(2\theta).$$

Thus,

$$\frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta = \frac{1}{2} \left[\frac{1}{2} \theta \sin(2\theta) - \frac{1}{2} \int \sin(2\theta) \, d\theta \right]_0^{\pi/2}.$$

Since $\sin(\pi) = 0$ and $\sin(0) = 0$, the boundary term vanishes. Also,

$$\int \sin(2\theta) \, d\theta = -\frac{1}{2} \cos(2\theta).$$

Therefore,

$$\frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta = \frac{1}{8} [\cos(2\theta)]_0^{\pi/2} = \frac{1}{8} (-1 - 1) = -\frac{1}{4}.$$

Combine results:

$$\frac{\pi^2}{16} - \frac{1}{4}.$$

$$\boxed{\frac{\pi^2}{16} - \frac{1}{4}}$$