

Question 1

Evaluate the definite integral

$$\int_0^1 (x^2 e^x - 2x e^x) \, dx.$$

Question 1 - Solution

Combine the terms:

$$\int_0^1 e^x(x^2 - 2x) dx.$$

Use integration by parts. Let

$$u = x^2 - 2x, \quad dv = e^x dx.$$

Then

$$du = (2x - 2) dx, \quad v = e^x.$$

Apply integration by parts:

$$\int e^x(x^2 - 2x) dx = (x^2 - 2x)e^x - \int e^x(2x - 2) dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = 2x - 2, \quad dv = e^x dx.$$

Then

$$du = 2 dx, \quad v = e^x.$$

Thus,

$$\int e^x(2x - 2) dx = (2x - 2)e^x - \int 2e^x dx = (2x - 2)e^x - 2e^x.$$

Substitute back:

$$\int e^x(x^2 - 2x) dx = (x^2 - 2x)e^x - [(2x - 2)e^x - 2e^x].$$

Simplify:

$$(x^2 - 4x + 4)e^x.$$

Now evaluate from 0 to 1:

$$\int_0^1 e^x(x^2 - 2x) dx = [(x^2 - 4x + 4)e^x]_0^1.$$

Compute:

$$(1 - 4 + 4)e - (0 - 0 + 4) = e - 4.$$

$$\boxed{e - 4}$$

Question 2

Evaluate the definite integral

$$\int_0^{\pi/2} (\sin x + \sin^2 x) \, dx.$$

Question 2 - Solution

Split the integral:

$$\int_0^{\pi/2} \sin x \, dx + \int_0^{\pi/2} \sin^2 x \, dx.$$

Evaluate the first integral:

$$\int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1.$$

For the second integral, use the identity

$$\sin^2 x = \frac{1 - \cos(2x)}{2}.$$

Then

$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx.$$

Integrate:

$$\frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi/2} = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4}.$$

Combine results:

$$1 + \frac{\pi}{4}.$$

$$\boxed{1 + \frac{\pi}{4}}$$

Question 3

Evaluate the definite integral

$$\int_{-1}^1 x^2 \sqrt{1-x^2} \, dx.$$

Question 3 - Solution

First notice a symmetry property. The function

$$x^2\sqrt{1-x^2}$$

is an *even function* because replacing x by $-x$ does not change its value. Therefore,

$$\int_{-1}^1 x^2\sqrt{1-x^2} dx = 2 \int_0^1 x^2\sqrt{1-x^2} dx.$$

Now focus on the integral from 0 to 1.

Use a trigonometric substitution. Let

$$x = \sin \theta, \quad dx = \cos \theta d\theta.$$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_0^1 x^2\sqrt{1-x^2} dx = \int_0^{\pi/2} (\sin^2 \theta)(\cos \theta)(\cos \theta) d\theta = \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta.$$

Use the identity

$$\sin^2 \theta \cos^2 \theta = \frac{1}{4} \sin^2(2\theta).$$

Then

$$\int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta = \frac{1}{4} \int_0^{\pi/2} \sin^2(2\theta) d\theta.$$

Apply the identity

$$\sin^2 u = \frac{1 - \cos(2u)}{2}.$$

Thus,

$$\frac{1}{4} \int_0^{\pi/2} \frac{1 - \cos(4\theta)}{2} d\theta = \frac{1}{8} \left[\theta - \frac{1}{4} \sin(4\theta) \right]_0^{\pi/2}.$$

Evaluate:

$$\frac{1}{8} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{16}.$$

Finally, multiply by 2 to account for the symmetry:

$$\int_{-1}^1 x^2\sqrt{1-x^2} dx = 2 \cdot \frac{\pi}{16} = \boxed{\frac{\pi}{8}}.$$

Question 4

Evaluate the definite integral

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} dx.$$

Question 4 - Solution

This integral is well-suited for a direct substitution because the derivative of $1+x^2$ appears in the numerator.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx \quad \Rightarrow \quad x \, dx = \frac{1}{2} \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{x}{\sqrt{1+x^2}} \, dx = \frac{1}{2} \int_1^2 u^{-1/2} \, du.$$

Integrate:

$$\frac{1}{2} \int u^{-1/2} \, du = \frac{1}{2} \cdot 2u^{1/2} = \sqrt{u}.$$

Apply the limits:

$$\left. \sqrt{u} \right|_1^2 = \sqrt{2} - 1.$$

$\sqrt{2} - 1$

Question 5

Evaluate the definite integral

$$\int_0^{\pi} x \sin x \, dx.$$

Question 5 - Solution

This integral suggests integration by parts because it is a product of x and $\sin x$.

Let

$$u = x, \quad dv = \sin x \, dx.$$

Then

$$du = dx, \quad v = -\cos x.$$

Apply integration by parts:

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x.$$

Now evaluate from 0 to π :

$$[-x \cos x + \sin x]_0^\pi.$$

Compute each endpoint:

$$-\pi \cos \pi + \sin \pi = \pi,$$

$$-0 \cdot \cos 0 + \sin 0 = 0.$$

Subtract:

$$\pi - 0 = \pi.$$

$$\boxed{\pi}$$

Question 6

Evaluate the definite integral

$$\int_{-2}^2 \frac{1}{1+x^2} dx.$$

Question 6 - Solution

First observe that the function

$$\frac{1}{1+x^2}$$

is an even function, since

$$\frac{1}{1+(-x)^2} = \frac{1}{1+x^2}.$$

Therefore, the integral over the symmetric interval $[-2, 2]$ can be written as

$$\int_{-2}^2 \frac{1}{1+x^2} dx = 2 \int_0^2 \frac{1}{1+x^2} dx.$$

Now evaluate the integral. Recall that

$$\int \frac{1}{1+x^2} dx = \arctan x.$$

Thus,

$$2 \int_0^2 \frac{1}{1+x^2} dx = 2 [\arctan x]_0^2.$$

Evaluate the bounds:

$$\arctan(2) - \arctan(0) = \arctan(2).$$

Therefore,

$$\int_{-2}^2 \frac{1}{1+x^2} dx = \boxed{2 \arctan(2)}.$$

Question 7

Evaluate the definite integral

$$\int_0^1 (3x^2 + 2)\sqrt{x} \, dx.$$

Question 7 - Solution

Begin by rewriting the integrand using powers of x :

$$(3x^2 + 2)\sqrt{x} = 3x^{5/2} + 2x^{1/2}.$$

Now integrate term by term:

$$\int 3x^{5/2} dx = 3 \cdot \frac{x^{7/2}}{7/2} = \frac{6}{7}x^{7/2},$$

$$\int 2x^{1/2} dx = 2 \cdot \frac{x^{3/2}}{3/2} = \frac{4}{3}x^{3/2}.$$

Combine the results:

$$\int (3x^2 + 2)\sqrt{x} dx = \frac{6}{7}x^{7/2} + \frac{4}{3}x^{3/2}.$$

Evaluate from 0 to 1:

$$\left[\frac{6}{7}x^{7/2} + \frac{4}{3}x^{3/2} \right]_0^1 = \frac{6}{7} + \frac{4}{3}.$$

Find a common denominator:

$$\frac{6}{7} + \frac{4}{3} = \frac{18 + 28}{21} = \frac{46}{21}.$$

$$\boxed{\frac{46}{21}}$$

Question 8

Evaluate the definite integral

$$\int_0^{\ln 2} e^x (1 + e^x) dx.$$

Question 8 - Solution

Begin by expanding the integrand:

$$e^x(1 + e^x) = e^x + e^{2x}.$$

Now integrate term by term:

$$\int e^x dx = e^x, \quad \int e^{2x} dx = \frac{1}{2}e^{2x}.$$

Thus,

$$\int e^x(1 + e^x) dx = e^x + \frac{1}{2}e^{2x}.$$

Evaluate from 0 to $\ln 2$:

$$\left[e^x + \frac{1}{2}e^{2x} \right]_0^{\ln 2}.$$

Compute the upper bound:

$$e^{\ln 2} + \frac{1}{2}e^{2 \ln 2} = 2 + \frac{1}{2} \cdot 4 = 4.$$

Compute the lower bound:

$$e^0 + \frac{1}{2}e^0 = 1 + \frac{1}{2} = \frac{3}{2}.$$

Subtract:

$$4 - \frac{3}{2} = \frac{5}{2}.$$

$$\boxed{\frac{5}{2}}$$

Question 9

Evaluate the definite integral

$$\int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{1}{x} \right) dx.$$

Question 9 - Solution

Rewrite the integrand using exponents:

$$\frac{2}{\sqrt{x}} - \frac{1}{x} = 2x^{-1/2} - x^{-1}.$$

Integrate term by term:

$$\int 2x^{-1/2} dx = 2 \cdot \frac{x^{1/2}}{1/2} = 4\sqrt{x},$$

$$\int x^{-1} dx = \ln x.$$

Thus,

$$\int \left(\frac{2}{\sqrt{x}} - \frac{1}{x} \right) dx = 4\sqrt{x} - \ln x.$$

Evaluate from 1 to 4:

$$\left[4\sqrt{x} - \ln x \right]_1^4.$$

Compute the upper bound:

$$4\sqrt{4} - \ln 4 = 8 - \ln 4.$$

Compute the lower bound:

$$4\sqrt{1} - \ln 1 = 4.$$

Subtract:

$$(8 - \ln 4) - 4 = 4 - \ln 4.$$

$$\boxed{4 - \ln 4}$$

Question 10

Evaluate the definite integral

$$\int_0^1 \frac{2x}{1+x^2} dx.$$

Question 10 - Solution

This integral suggests a substitution because the derivative of the denominator appears in the numerator.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x \, dx.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute into the integral:

$$\int_0^1 \frac{2x}{1+x^2} \, dx = \int_1^2 \frac{1}{u} \, du.$$

Integrate:

$$\int \frac{1}{u} \, du = \ln u.$$

Apply the limits:

$$\ln u \Big|_1^2 = \ln 2.$$

$$\boxed{\ln 2}$$