

Question 1

Consider the function

$$f(x) = 2x + 1$$

on the interval $[0, 2]$.

Using the definition of the definite integral, express

$$\int_0^2 (2x + 1) \, dx$$

as the limit of a Riemann sum with n equal subintervals.

Question 1 - Solution

Divide the interval $[0, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = 0 + i\Delta x = \frac{2i}{n}.$$

Evaluate the function at x_i :

$$f(x_i) = 2\left(\frac{2i}{n}\right) + 1 = \frac{4i}{n} + 1.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(\frac{4i}{n} + 1\right) \frac{2}{n}.$$

Simplify:

$$\frac{2}{n} \sum_{i=1}^n \left(\frac{4i}{n} + 1\right) = \frac{8}{n^2} \sum_{i=1}^n i + \frac{2}{n} \sum_{i=1}^n 1.$$

Using

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n 1 = n,$$

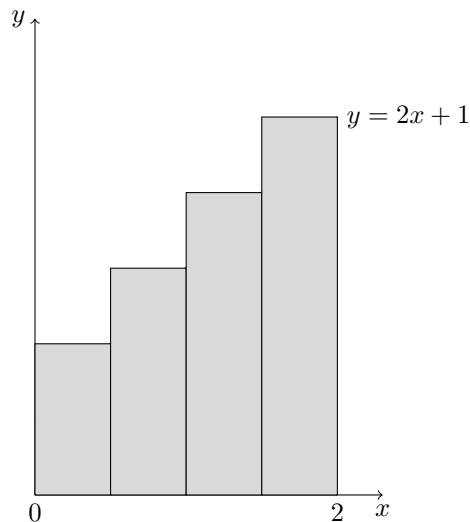
we obtain

$$\frac{8}{n^2} \cdot \frac{n(n+1)}{2} + \frac{2}{n} \cdot n = 4\left(1 + \frac{1}{n}\right) + 2.$$

Thus,

$$\int_0^2 (2x + 1) dx = \lim_{n \rightarrow \infty} \left(6 + \frac{4}{n}\right) = 6.$$

6



Question 2

Let

$$f(x) = x^2$$

on the interval $[1, 3]$.

Using the definition of the definite integral, write

$$\int_1^3 x^2 dx$$

as the limit of a Riemann sum using midpoints.

Question 2 - Solution

Divide $[1, 3]$ into n equal subintervals of width $\Delta x = 2/n$.

The midpoints are $x_i^* = 1 + (2i - 1)/n$.

$$M_n = \frac{2}{n} \sum_{i=1}^n \left(1 + \frac{2i-1}{n} \right)^2.$$

Use $\sum (2i - 1) = n^2$ and $\sum (2i - 1)^2 = n(4n^2 - 1)/3$:

$$M_n = 2 + 4 + \frac{2(4n^2 - 1)}{3n^2} = \frac{26}{3} - \frac{2}{3n^2}.$$

$$\boxed{\int_1^3 x^2 dx = \lim_{n \rightarrow \infty} M_n = \frac{26}{3}.$$

Question 3

Let

$$f(x) = \sqrt{x}$$

on the interval $[0, 4]$.

Using the definition of the definite integral, write

$$\int_0^4 \sqrt{x} \, dx$$

as the limit of a Riemann sum using left endpoints, and evaluate the limit.

Question 3 - Solution

Divide the interval $[0, 4]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{4}{n}.$$

Using left endpoints, the i th sample point is

$$x_{i-1} = (i-1)\Delta x = \frac{4(i-1)}{n}.$$

Evaluate the function:

$$f(x_{i-1}) = \sqrt{\frac{4(i-1)}{n}} = 2\sqrt{\frac{i-1}{n}}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_{i-1}) \Delta x = \sum_{i=1}^n 2\sqrt{\frac{i-1}{n}} \cdot \frac{4}{n} = \frac{8}{n^{3/2}} \sum_{i=1}^n \sqrt{i-1}.$$

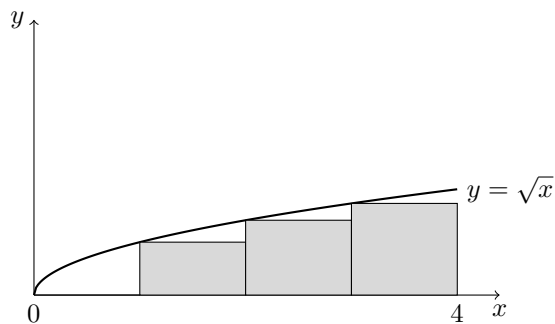
Thus,

$$\int_0^4 \sqrt{x} \, dx = \lim_{n \rightarrow \infty} \frac{8}{n^{3/2}} \sum_{i=1}^n \sqrt{i-1}.$$

As $n \rightarrow \infty$, this Riemann sum converges to the definite integral:

$$\int_0^4 \sqrt{x} \, dx = \left[\frac{2}{3} x^{3/2} \right]_0^4 = \frac{16}{3}.$$

$$\boxed{\frac{16}{3}}$$



Question 4

Let

$$f(x) = \frac{1}{1+x^2}$$

on the interval $[-1, 1]$.

Using the definition of the definite integral, express

$$\int_{-1}^1 \frac{1}{1+x^2} dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

Question 4 - Solution

Divide the interval $[-1, 1]$ into n equal subintervals. The width of each subinterval is

$$\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = -1 + i\Delta x = -1 + \frac{2i}{n}.$$

Evaluate the function:

$$f(x_i) = \frac{1}{1 + \left(-1 + \frac{2i}{n}\right)^2}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{2}{n} \sum_{i=1}^n \frac{1}{1 + \left(-1 + \frac{2i}{n}\right)^2}.$$

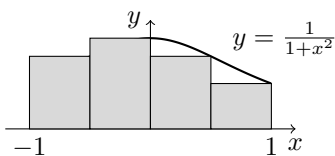
Thus,

$$\int_{-1}^1 \frac{1}{1+x^2} dx = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \frac{1}{1 + \left(-1 + \frac{2i}{n}\right)^2}.$$

Evaluating the definite integral,

$$\int_{-1}^1 \frac{1}{1+x^2} dx = [\arctan x]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

$$\boxed{\frac{\pi}{2}}$$



Question 5

Let

$$f(x) = e^x$$

on the interval $[0, 1]$.

Using the definition of the definite integral, express

$$\int_0^1 e^x dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

Question 5- Solution

Partition the interval $[0, 1]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{1}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = i\Delta x = \frac{i}{n}.$$

Evaluate the function at each sample point:

$$f(x_i) = e^{i/n}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{1}{n} \sum_{i=1}^n e^{i/n}.$$

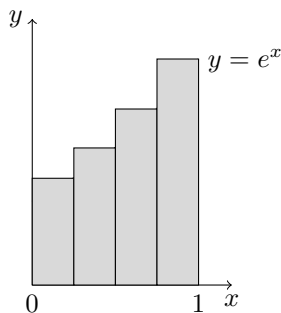
Thus,

$$\int_0^1 e^x dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n e^{i/n}.$$

Evaluating the definite integral,

$$\int_0^1 e^x dx = [e^x]_0^1 = e - 1.$$

$e - 1$



Question 6

Let

$$f(x) = |x|$$

on the interval $[-2, 2]$.

Using the definition of the definite integral, express

$$\int_{-2}^2 |x| dx$$

as the limit of a Riemann sum using equal subintervals, and evaluate the limit.

Question 6 - Solution

Divide the interval $[-2, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2 - (-2)}{n} = \frac{4}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = -2 + i\Delta x = -2 + \frac{4i}{n}.$$

Evaluate the function:

$$f(x_i) = \left| -2 + \frac{4i}{n} \right|.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{4}{n} \sum_{i=1}^n \left| -2 + \frac{4i}{n} \right|.$$

Thus,

$$\int_{-2}^2 |x| dx = \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left| -2 + \frac{4i}{n} \right|.$$

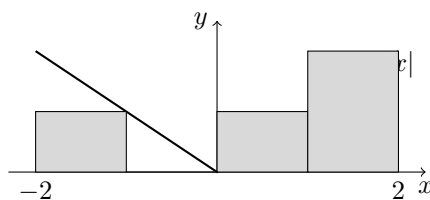
To evaluate the integral, use symmetry:

$$\int_{-2}^2 |x| dx = 2 \int_0^2 x dx.$$

Compute:

$$2 \left[\frac{x^2}{2} \right]_0^2 = 4.$$

$$\boxed{4}$$



Question 7

Let

$$f(x) = 3 - 2x$$

on the interval $[0, 1]$.

Using the definition of the definite integral, express

$$\int_0^1 (3 - 2x) \, dx$$

as the limit of a Riemann sum using left endpoints, and evaluate the limit.

Question 7 - Solution

Partition the interval $[0, 1]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{1}{n}.$$

Using left endpoints, the i th sample point is

$$x_{i-1} = (i-1)\Delta x = \frac{i-1}{n}.$$

Evaluate the function at each sample point:

$$f(x_{i-1}) = 3 - 2\left(\frac{i-1}{n}\right).$$

The Riemann sum is

$$\sum_{i=1}^n f(x_{i-1}) \Delta x = \frac{1}{n} \sum_{i=1}^n \left(3 - \frac{2(i-1)}{n}\right).$$

Separate the sum:

$$\frac{1}{n} \left(\sum_{i=1}^n 3 - \frac{2}{n} \sum_{i=1}^n (i-1) \right).$$

Using

$$\sum_{i=1}^n 1 = n, \quad \sum_{i=1}^n (i-1) = \frac{n(n-1)}{2},$$

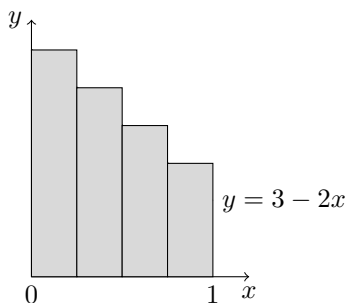
we obtain

$$\frac{1}{n} \left(3n - \frac{2}{n} \cdot \frac{n(n-1)}{2} \right) = 3 - (n-1)\frac{1}{n}.$$

Thus,

$$\int_0^1 (3-2x) dx = \lim_{n \rightarrow \infty} \left(3 - \frac{n-1}{n} \right) = 2.$$

2



Question 8

Let

$$f(x) = \frac{1}{x}$$

on the interval $[1, e]$.

Using the definition of the definite integral, express

$$\int_1^e \frac{1}{x} dx$$

as the limit of a Riemann sum using right endpoints, and evaluate the limit.

Question 8 - Solution

Partition the interval $[1, e]$ into n equal subintervals. The width of each subinterval is

$$\Delta x = \frac{e - 1}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = 1 + i\Delta x = 1 + i\frac{e - 1}{n}.$$

Evaluate the function at each sample point:

$$f(x_i) = \frac{1}{1 + i\frac{e-1}{n}}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{e - 1}{n} \sum_{i=1}^n \frac{1}{1 + i\frac{e-1}{n}}.$$

Thus,

$$\int_1^e \frac{1}{x} dx = \lim_{n \rightarrow \infty} \frac{e - 1}{n} \sum_{i=1}^n \frac{1}{1 + i\frac{e-1}{n}}.$$

Evaluating the definite integral,

$$\int_1^e \frac{1}{x} dx = [\ln x]_1^e = 1.$$

1

Question 9

Let

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2. \end{cases}$$

Using the definition of the definite integral, express

$$\int_0^2 f(x) \, dx$$

as the limit of a Riemann sum, and evaluate the limit.

Question 9 - Solution

Partition the interval $[0, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = \frac{2i}{n}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{2}{n} \sum_{i=1}^n f\left(\frac{2i}{n}\right).$$

Split the sum according to where the function changes behavior. The condition $x_i \leq 1$ corresponds to

$$\frac{2i}{n} \leq 1 \quad \Rightarrow \quad i \leq \frac{n}{2}.$$

Thus,

$$\sum_{i=1}^n f\left(\frac{2i}{n}\right) = \sum_{i=1}^{n/2} 2\left(\frac{2i}{n}\right) + \sum_{i=n/2+1}^n 2.$$

Compute each sum:

$$\sum_{i=1}^{n/2} \frac{4i}{n} = \frac{4}{n} \cdot \frac{(n/2)(n/2+1)}{2} = \frac{n}{2} + 1,$$

$$\sum_{i=n/2+1}^n 2 = 2 \cdot \frac{n}{2} = n.$$

Therefore,

$$\sum_{i=1}^n f\left(\frac{2i}{n}\right) = \frac{n}{2} + 1 + n = \frac{3n}{2} + 1.$$

Multiply by Δx :

$$\frac{2}{n} \left(\frac{3n}{2} + 1 \right) = 3 + \frac{2}{n}.$$

Taking the limit,

$$\int_0^2 f(x) dx = \lim_{n \rightarrow \infty} \left(3 + \frac{2}{n} \right) = 3.$$

Question 10

Let

$$f(x) = \sin x$$

on the interval $[0, \pi]$.

Using the definition of the definite integral, express

$$\int_0^{\pi} \sin x \, dx$$

as the limit of a Riemann sum using midpoints, and evaluate the limit.

Question 10 - Solution

Partition the interval $[0, \pi]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{\pi}{n}.$$

The midpoint of the i th subinterval is

$$x_i^* = \left(i - \frac{1}{2}\right) \Delta x = \left(i - \frac{1}{2}\right) \frac{\pi}{n}.$$

Evaluate the function at each midpoint:

$$f(x_i^*) = \sin\left(\left(i - \frac{1}{2}\right) \frac{\pi}{n}\right).$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i^*) \Delta x = \frac{\pi}{n} \sum_{i=1}^n \sin\left(\left(i - \frac{1}{2}\right) \frac{\pi}{n}\right).$$

Thus,

$$\int_0^{\pi} \sin x \, dx = \lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{i=1}^n \sin\left(\left(i - \frac{1}{2}\right) \frac{\pi}{n}\right).$$

Evaluating the definite integral directly,

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2.$$

2