

Question 1

Find the exact area of the region enclosed by the curves

$$y = x\sqrt{4 - x^2} \quad \text{and} \quad y = x^2.$$

Question 1 - Solution

First find the points of intersection:

$$x\sqrt{4-x^2} = x^2.$$

Factor:

$$x(\sqrt{4-x^2} - x) = 0.$$

Thus,

$$x = 0 \quad \text{or} \quad \sqrt{4-x^2} = x.$$

Square the second equation:

$$4 - x^2 = x^2 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2}.$$

On the interval $[0, \sqrt{2}]$, we have

$$x\sqrt{4-x^2} \geq x^2.$$

Therefore, the area is

$$A = \int_0^{\sqrt{2}} (x\sqrt{4-x^2} - x^2) dx.$$

Split the integral:

$$A = \int_0^{\sqrt{2}} x\sqrt{4-x^2} dx - \int_0^{\sqrt{2}} x^2 dx.$$

For the first integral, use substitution. Let

$$u = 4 - x^2, \quad du = -2x dx.$$

Then

$$\int x\sqrt{4-x^2} dx = -\frac{1}{2} \int u^{1/2} du = -\frac{1}{3} u^{3/2} = -\frac{1}{3} (4-x^2)^{3/2}.$$

Evaluate from 0 to $\sqrt{2}$:

$$\left[-\frac{1}{3} (4-x^2)^{3/2} \right]_0^{\sqrt{2}} = -\frac{1}{3} (2\sqrt{2}) + \frac{1}{3} (8) = \frac{8-2\sqrt{2}}{3}.$$

For the second integral:

$$\int_0^{\sqrt{2}} x^2 dx = \left[\frac{x^3}{3} \right]_0^{\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

Subtract:

$$A = \frac{8-2\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} = \frac{8-4\sqrt{2}}{3}.$$

$$\boxed{\frac{8-4\sqrt{2}}{3}}$$

Question 2

Find the exact area of the region enclosed by the curves

$$y = \ln x \quad \text{and} \quad y = x - 1,$$

restricted to the interval $0 < x \leq 1$.

Question 2 - Solution

On the interval $0 < x \leq 1$, the curves intersect where

$$\ln x = x - 1,$$

which occurs at

$$x = 1.$$

For $0 < x < 1$, we have

$$x - 1 \geq \ln x,$$

so the area between the curves is

$$A = \int_0^1 ((x - 1) - \ln x) dx.$$

Compute each term:

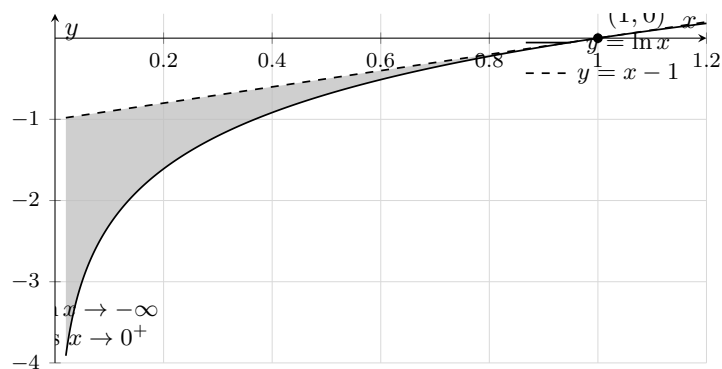
$$\int_0^1 (x - 1) dx = \left[\frac{x^2}{2} - x \right]_0^1 = -\frac{1}{2},$$

$$\int_0^1 \ln x dx = [x \ln x - x]_0^1 = -1.$$

Therefore,

$$A = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$



Question 3

Find the exact area of the region enclosed by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = x^2,$$

restricted to the interval $0 \leq x \leq 1$.

Question 3 - Solution

First determine which curve lies above the other on $[0, 1]$. For $0 < x < 1$,

$$\sqrt{x} \geq x^2.$$

Thus, the enclosed area is

$$A = \int_0^1 (\sqrt{x} - x^2) dx.$$

Evaluate each term:

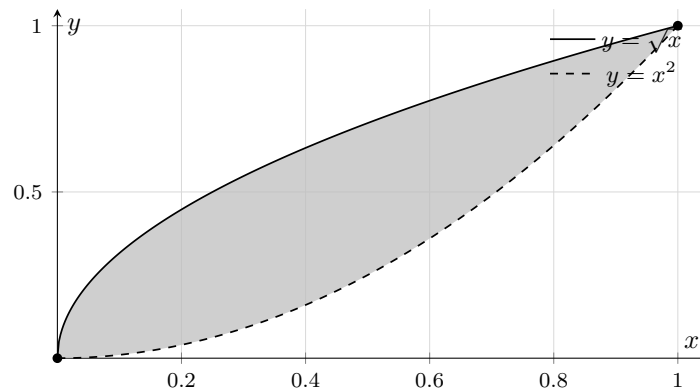
$$\int_0^1 \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_0^1 = \frac{2}{3},$$

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$

Subtract:

$$A = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

$$\boxed{\frac{1}{3}}$$



Question 4

Find the exact area of the region enclosed by the curves

$$y = x\sqrt{2-x} \quad \text{and} \quad y = x.$$

Question 4 - Solution

The intersections satisfy $x\sqrt{2-x} = x$, giving $x = 0, 1$.

For $0 < x < 1$, the square-root curve is above the line, hence

$$A = \int_0^1 (x\sqrt{2-x} - x) dx.$$

With $u = 2 - x$,

$$\int_0^1 x\sqrt{2-x} dx = \int_1^2 (2-u)\sqrt{u} du = \left[\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right]_1^2 = \frac{16\sqrt{2} - 14}{15}.$$

$$A = \frac{32\sqrt{2} - 43}{30} > 0.$$

Question 5

Find the exact area of the region enclosed by the curves

$$y = |x| \quad \text{and} \quad y = x^2.$$

Question 5 - Solution

First find the points of intersection by solving

$$|x| = x^2.$$

For $x \geq 0$, this becomes

$$x = x^2 \Rightarrow x = 0 \text{ or } x = 1.$$

For $x < 0$, we have

$$-x = x^2 \Rightarrow x = 0 \text{ or } x = -1.$$

Thus, the curves intersect at

$$x = -1, 0, 1.$$

On $[0, 1]$, we have $|x| = x \geq x^2$, and by symmetry the same holds on $[-1, 0]$. Therefore, the total area is twice the area on $[0, 1]$:

$$A = 2 \int_0^1 (x - x^2) dx.$$

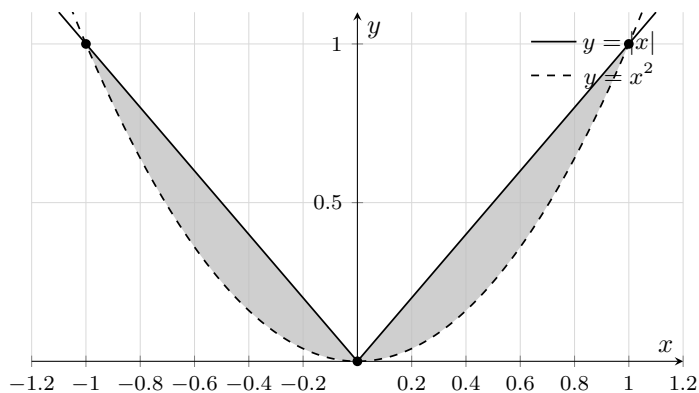
Compute the integral:

$$\int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Thus,

$$A = 2 \cdot \frac{1}{6} = \frac{1}{3}.$$

$$\boxed{\frac{1}{3}}$$



Question 6

Find the exact area of the region enclosed by the curves

$$y = x \quad \text{and} \quad y = x^3.$$

Question 6 - Solution

First find the points of intersection:

$$x = x^3 \Rightarrow x(x^2 - 1) = 0.$$

Thus,

$$x = -1, 0, 1.$$

On the interval $(0, 1)$, we have

$$x \geq x^3,$$

and by symmetry, the same configuration occurs on $(-1, 0)$.

Therefore, the total enclosed area is twice the area on $[0, 1]$:

$$A = 2 \int_0^1 (x - x^3) dx.$$

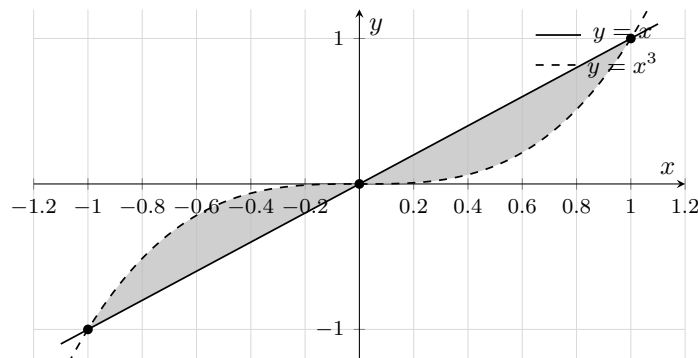
Evaluate the integral:

$$\int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Thus,

$$A = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

$$\boxed{\frac{1}{2}}$$



Question 7

Find the exact area of the region enclosed by the curves

$$y = \sin x \quad \text{and} \quad y = \cos x$$

on the interval

$$\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}.$$

Question 7 - Solution

First find the points of intersection by solving

$$\sin x = \cos x.$$

This gives

$$\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} + k\pi.$$

The smallest two consecutive intersection points occur at

$$x = \frac{\pi}{4} \quad \text{and} \quad x = \frac{5\pi}{4}.$$

On the interval between these points, we have

$$\sin x \geq \cos x.$$

Thus the enclosed area is

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx.$$

Compute:

$$\int \sin x dx = -\cos x, \quad \int \cos x dx = \sin x.$$

So

$$A = [-\cos x - \sin x]_{\pi/4}^{5\pi/4}.$$

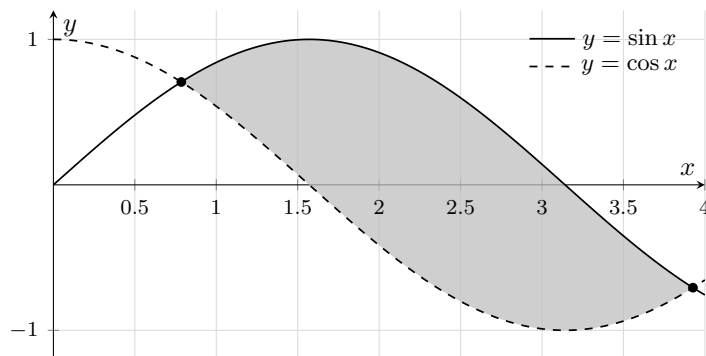
Evaluate:

$$\begin{aligned} -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) &= \sqrt{2}, \\ -\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) &= -\sqrt{2}. \end{aligned}$$

Thus,

$$A = \sqrt{2} - (-\sqrt{2}) = 2\sqrt{2}.$$

$$\boxed{2\sqrt{2}}$$



Question 8

Find the exact area of the region enclosed by the curves

$$y = e^{-x} \quad \text{and} \quad y = 1 + \left(\frac{1}{e} - 1\right)x.$$

Question 8 - Solution

First find the intersection points:

$$e^{-x} = 1 + \left(\frac{1}{e} - 1\right)x.$$

At $x = 0$,

$$e^0 = 1 = 1 + \left(\frac{1}{e} - 1\right) \cdot 0.$$

At $x = 1$,

$$e^{-1} = \frac{1}{e} = 1 + \left(\frac{1}{e} - 1\right) \cdot 1.$$

So the curves intersect at $x = 0$ and $x = 1$.

Since e^{-x} is convex, the chord (the line) lies above e^{-x} on $(0, 1)$, so the area is

$$A = \int_0^1 \left(1 + \left(\frac{1}{e} - 1\right)x - e^{-x}\right) dx.$$

Compute each piece:

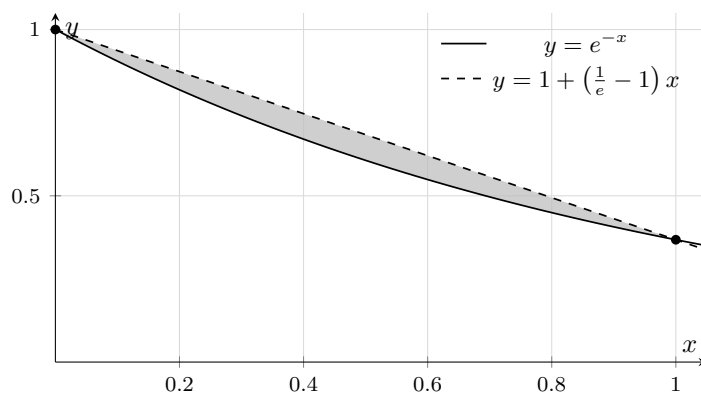
$$\int_0^1 \left(1 + \left(\frac{1}{e} - 1\right)x\right) dx = \left[x + \left(\frac{1}{e} - 1\right)\frac{x^2}{2}\right]_0^1 = 1 + \frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2} + \frac{1}{2e},$$

$$\int_0^1 e^{-x} dx = [-e^{-x}]_0^1 = 1 - \frac{1}{e}.$$

Thus

$$A = \left(\frac{1}{2} + \frac{1}{2e}\right) - \left(1 - \frac{1}{e}\right) = -\frac{1}{2} + \frac{3}{2e} = \frac{1}{2}\left(\frac{3}{e} - 1\right).$$

$$\boxed{\frac{1}{2}\left(\frac{3}{e} - 1\right)}$$



Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1 + x) \quad \text{and} \quad y = \frac{x}{1 + x},$$

between $x = 0$ and $x = 1$.

Question 9 - Solution

On the interval $0 \leq x \leq 1$, we compare the curves. At $x = 0$,

$$\ln(1) = 0, \quad \frac{0}{1} = 0.$$

At $x = 1$,

$$\ln 2 > \frac{1}{2},$$

so

$$\ln(1+x) \geq \frac{x}{1+x} \quad \text{on } [0, 1].$$

Thus the enclosed area is

$$A = \int_0^1 \left(\ln(1+x) - \frac{x}{1+x} \right) dx.$$

Split the integral:

$$A = \int_0^1 \ln(1+x) dx - \int_0^1 \frac{x}{1+x} dx.$$

Evaluate the first integral:

$$\int \ln(1+x) dx = (1+x) \ln(1+x) - x,$$

so

$$\int_0^1 \ln(1+x) dx = [(1+x) \ln(1+x) - x]_0^1 = 2 \ln 2 - 1.$$

For the second integral, rewrite:

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

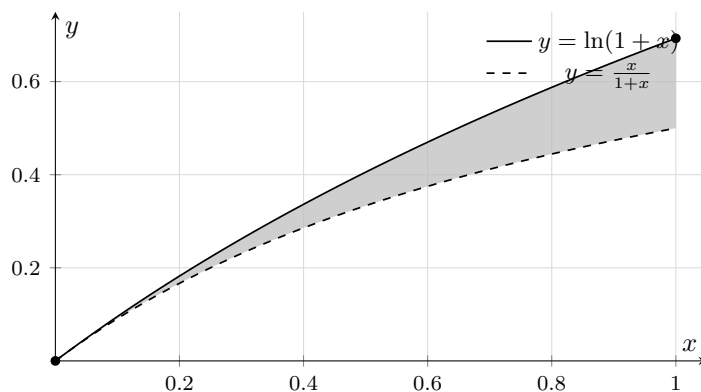
Then

$$\int_0^1 \frac{x}{1+x} dx = \int_0^1 \left(1 - \frac{1}{1+x} \right) dx = 1 - \ln 2.$$

Subtract:

$$A = (2 \ln 2 - 1) - (1 - \ln 2) = 3 \ln 2 - 2.$$

$$\boxed{3 \ln 2 - 2}$$



Question 10

Find the exact area of the region bounded by the curves

$$y = \sqrt{x} \quad \text{and} \quad y = \ln x,$$

between $x = 1$ and $x = e^2$.

Question 10 - Solution

On $1 \leq x \leq e^2$, we have $\sqrt{x} \geq \ln x$, so

$$A = \int_1^{e^2} (\sqrt{x} - \ln x) dx.$$

Compute:

$$\int_1^{e^2} \sqrt{x} dx = \left[\frac{2}{3} x^{3/2} \right]_1^{e^2} = \frac{2}{3} (e^3 - 1),$$

$$\int_1^{e^2} \ln x dx = [x \ln x - x]_1^{e^2} = e^2 + 1.$$

Thus

$$A = \frac{2}{3} (e^3 - 1) - (e^2 + 1) = \frac{2}{3} e^3 - e^2 - \frac{5}{3}.$$

$$\boxed{\frac{2}{3} e^3 - e^2 - \frac{5}{3}}$$

