

Question 1

Evaluate the integral

$$\int \frac{x}{1+x^2} \sqrt{1+\ln(1+x^2)} \, dx.$$

Question 1 - Solution**Step 1: Choose a substitution**

Set $u = 1 + \ln(1 + x^2)$, so $du = 2x/(1 + x^2) dx$.

$$\int \frac{x}{1 + x^2} \sqrt{1 + \ln(1 + x^2)} dx = \frac{1}{2} \int u^{1/2} du = \frac{1}{3} u^{3/2} + C.$$

$$\frac{1}{3} (1 + \ln(1 + x^2))^{3/2} + C.$$

This is valid for every real x .

Question 2

Evaluate the integral

$$\int \frac{x^2}{\sqrt{4-x^2}} dx.$$

Question 2 - Solution

Because the integrand contains $\sqrt{4-x^2}$, a trigonometric substitution is natural.

Let

$$x = 2 \sin \theta.$$

Then

$$dx = 2 \cos \theta \, d\theta, \quad \sqrt{4-x^2} = \sqrt{4-4\sin^2 \theta} = 2 \cos \theta.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{4-x^2}} dx = \int \frac{4 \sin^2 \theta}{2 \cos \theta} \cdot 2 \cos \theta \, d\theta = \int 4 \sin^2 \theta \, d\theta.$$

Use the identity

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}.$$

Then

$$\int 4 \sin^2 \theta \, d\theta = 4 \int \frac{1 - \cos(2\theta)}{2} d\theta = 2 \int (1 - \cos(2\theta)) \, d\theta.$$

Integrate:

$$2 \left(\theta - \frac{1}{2} \sin(2\theta) \right) = 2\theta - \sin(2\theta).$$

Return to x using

$$\theta = \arcsin\left(\frac{x}{2}\right), \quad \sin(2\theta) = 2 \sin \theta \cos \theta = \frac{x}{2} \sqrt{4-x^2}.$$

Thus,

$$2 \arcsin\left(\frac{x}{2}\right) - \frac{x}{2} \sqrt{4-x^2} + C$$

Question 3

Evaluate the integral

$$\int \frac{x^2}{\sqrt{x^2 - 9}} dx, \quad x > 3.$$

Question 3 - Solution

Because the integrand involves $\sqrt{x^2 - a^2}$, a *hyperbolic substitution* is natural.

Let

$$x = 3 \cosh t.$$

Then

$$dx = 3 \sinh t \, dt, \quad \sqrt{x^2 - 9} = \sqrt{9 \cosh^2 t - 9} = 3 \sinh t.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{x^2 - 9}} dx = \int \frac{9 \cosh^2 t}{3 \sinh t} \cdot 3 \sinh t \, dt = \int 9 \cosh^2 t \, dt.$$

Use the identity

$$\cosh^2 t = \frac{1 + \cosh(2t)}{2}.$$

Then

$$\int 9 \cosh^2 t \, dt = \frac{9}{2} \int (1 + \cosh(2t)) \, dt = \frac{9}{2} \left(t + \frac{1}{2} \sinh(2t) \right).$$

Simplify:

$$\frac{9}{2}t + \frac{9}{4} \sinh(2t).$$

Return to x . Since

$$x = 3 \cosh t \quad \Rightarrow \quad t = \operatorname{arcosh}\left(\frac{x}{3}\right),$$

and

$$\sinh(2t) = 2 \sinh t \cosh t = 2 \left(\frac{\sqrt{x^2 - 9}}{3} \right) \left(\frac{x}{3} \right) = \frac{2x\sqrt{x^2 - 9}}{9}.$$

Substitute back:

$$\boxed{\frac{9}{2} \operatorname{arcosh}\left(\frac{x}{3}\right) + \frac{x}{2} \sqrt{x^2 - 9} + C}$$

Question 4

Evaluate the integral

$$\int \frac{x}{\sqrt{x^2 + 2x + 5}} dx.$$

Question 4 - Solution

Complete the square:

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Let

$$u = x + 1, \quad du = dx, \quad x = u - 1.$$

Then the integral becomes

$$\int \frac{u-1}{\sqrt{u^2+4}} du = \int \frac{u}{\sqrt{u^2+4}} du - \int \frac{1}{\sqrt{u^2+4}} du.$$

First integral: Let $w = u^2 + 4$, so $dw = 2u du$:

$$\int \frac{u}{\sqrt{u^2+4}} du = \frac{1}{2} \int w^{-1/2} dw = \sqrt{u^2+4}.$$

Second integral (expanded):

$$\int \frac{1}{\sqrt{u^2+4}} du.$$

Use a trigonometric substitution. Let

$$u = 2 \tan \theta.$$

Then

$$du = 2 \sec^2 \theta d\theta, \quad \sqrt{u^2+4} = \sqrt{4 \tan^2 \theta + 4} = 2 \sec \theta.$$

Substitute:

$$\int \frac{1}{\sqrt{u^2+4}} du = \int \frac{2 \sec^2 \theta}{2 \sec \theta} d\theta = \int \sec \theta d\theta.$$

Recall:

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Return to u :

$$\tan \theta = \frac{u}{2}, \quad \sec \theta = \frac{\sqrt{u^2+4}}{2}.$$

Thus,

$$\int \frac{1}{\sqrt{u^2+4}} du = \ln(u + \sqrt{u^2+4}).$$

Final Answer:

$$\boxed{\sqrt{x^2+2x+5} - \ln(x+1+\sqrt{x^2+2x+5}) + C}$$

Question 5

Evaluate the integral

$$\int \frac{dx}{1 + \sin x}.$$

Question 5 - Solution

The integrand is defined wherever $1 + \sin x \neq 0$.

A primitive valid on each such interval is

$$\boxed{-\frac{\cos x}{1 + \sin x} + C.}$$

Indeed, the quotient rule gives

$$\frac{d}{dx} \left(-\frac{\cos x}{1 + \sin x} \right) = \frac{\sin x(1 + \sin x) + \cos^2 x}{(1 + \sin x)^2} = \frac{1}{1 + \sin x}.$$

This expression introduces no extra excluded points.

Question 6

Evaluate the integral

$$\int \sqrt{x^2 + x} \, dx.$$

Question 6 - Solution

Because the integrand involves a square root of a quadratic expression, an *Euler substitution* is effective.

Complete the square:

$$x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Use the substitution

$$x + \frac{1}{2} = \frac{1}{2} \left(t - \frac{1}{t}\right), \quad t > 0.$$

Then

$$dx = \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt,$$

and

$$\sqrt{x^2 + x} = \sqrt{\frac{1}{4} \left(t - \frac{1}{t}\right)^2 - \frac{1}{4}} = \frac{1}{2} \left(t - \frac{1}{t}\right).$$

Substitute into the integral:

$$\int \sqrt{x^2 + x} dx = \int \frac{1}{2} \left(t - \frac{1}{t}\right) \cdot \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt.$$

Simplify:

$$\frac{1}{4} \int \left(t - \frac{1}{t^3}\right) dt.$$

Integrate:

$$\frac{1}{4} \left(\frac{t^2}{2} + \frac{1}{2t^2}\right) = \frac{1}{8} \left(t^2 + \frac{1}{t^2}\right).$$

Substitute back using

$$t - \frac{1}{t} = 2 \left(x + \frac{1}{2}\right).$$

Thus,

$$\boxed{\frac{1}{2} \left(x + \frac{1}{2}\right) \sqrt{x^2 + x} - \frac{1}{8} \ln \left|2x + 1 + 2\sqrt{x^2 + x}\right| + C}$$

Question 7

Evaluate the integral

$$\int \sqrt{x^2 + x} \, dx.$$

Question 7 - Solution

Begin by completing the square:

$$x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Let

$$u = x + \frac{1}{2}.$$

Then

$$du = dx, \quad \sqrt{x^2 + x} = \sqrt{u^2 - \frac{1}{4}}.$$

The integral becomes

$$\int \sqrt{u^2 - \frac{1}{4}} du.$$

Now use a standard substitution for expressions of the form $\sqrt{u^2 - a^2}$. Let

$$u = \frac{1}{2} \sec \theta.$$

Then

$$du = \frac{1}{2} \sec \theta \tan \theta d\theta, \quad \sqrt{u^2 - \frac{1}{4}} = \frac{1}{2} \tan \theta.$$

Substitute:

$$\int \sqrt{u^2 - \frac{1}{4}} du = \int \frac{1}{2} \tan \theta \cdot \frac{1}{2} \sec \theta \tan \theta d\theta = \frac{1}{4} \int \tan^2 \theta \sec \theta d\theta.$$

Use the identity $\tan^2 \theta = \sec^2 \theta - 1$:

$$\frac{1}{4} \int (\sec^3 \theta - \sec \theta) d\theta.$$

Integrate:

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|),$$

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Thus,

$$\frac{1}{4} \left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right].$$

Return to x using

$$\sec \theta = 2u = 2x + 1, \quad \tan \theta = 2\sqrt{x^2 + x}.$$

Final Answer:

$$\boxed{\frac{1}{2} \left(x + \frac{1}{2}\right) \sqrt{x^2 + x} - \frac{1}{8} \ln |2x + 1 + 2\sqrt{x^2 + x}| + C}$$

Question 8

Evaluate the integral

$$\int \frac{x+2}{\sqrt{x^2+4x+8}} dx.$$

Question 8 - Solution

Complete the square in the denominator:

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Let

$$u = x + 2.$$

Then

$$du = dx, \quad x + 2 = u.$$

Substitute into the integral:

$$\int \frac{x + 2}{\sqrt{x^2 + 4x + 8}} dx = \int \frac{u}{\sqrt{u^2 + 4}} du.$$

Now use a direct substitution. Let

$$w = u^2 + 4, \quad dw = 2u du.$$

Then

$$\int \frac{u}{\sqrt{u^2 + 4}} du = \frac{1}{2} \int w^{-1/2} dw = \sqrt{w}.$$

Substitute back:

$$\boxed{\sqrt{x^2 + 4x + 8} + C}$$

Question 9

Evaluate the integral

$$\int \frac{x+1}{(x^2+2x+3)^{3/2}} dx.$$

Question 9 - Solution

Begin by completing the square in the denominator:

$$x^2 + 2x + 3 = (x + 1)^2 + 2.$$

Let

$$u = x + 1.$$

Then

$$du = dx, \quad (x^2 + 2x + 3)^{3/2} = (u^2 + 2)^{3/2}.$$

Substitute into the integral:

$$\int \frac{u}{(u^2 + 2)^{3/2}} du.$$

Now use a direct substitution. Let

$$w = u^2 + 2, \quad dw = 2u du.$$

Then

$$\int \frac{u}{(u^2 + 2)^{3/2}} du = \frac{1}{2} \int w^{-3/2} dw.$$

Integrate:

$$\frac{1}{2} \int w^{-3/2} dw = \frac{1}{2} \left(-2w^{-1/2} \right) = -\frac{1}{\sqrt{w}}.$$

Substitute back:

$$\boxed{-\frac{1}{\sqrt{x^2 + 2x + 3}} + C}$$

Question 10

Evaluate the integral

$$\int \frac{e^x}{\sqrt{1 + e^{2x}}} dx.$$

Question 10 - Solution

Begin with an exponential substitution. Let

$$u = e^x.$$

Then

$$du = e^x dx.$$

Substitute into the integral:

$$\int \frac{e^x}{\sqrt{1+e^{2x}}} dx = \int \frac{1}{\sqrt{1+u^2}} du.$$

Inverse trigonometric step (expanded):

We recognize the integrand as the derivative of $\operatorname{arsinh} u$ or $\arctan u$ -type expressions. Use a trigonometric substitution to justify the result.

Let

$$u = \tan \theta.$$

Then

$$du = \sec^2 \theta d\theta, \quad \sqrt{1+u^2} = \sqrt{1+\tan^2 \theta} = \sec \theta.$$

Substitute:

$$\int \frac{1}{\sqrt{1+u^2}} du = \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \int \sec \theta d\theta.$$

Recall the standard integral:

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Substitute back using

$$\tan \theta = u, \quad \sec \theta = \sqrt{1+u^2}.$$

Thus,

$$\int \frac{1}{\sqrt{1+u^2}} du = \ln(u + \sqrt{1+u^2}).$$

Substitute back $u = e^x$:

$$\boxed{\ln(e^x + \sqrt{1+e^{2x}}) + C}$$