

Question 1

Evaluate the integral

$$\int \frac{x e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}} dx.$$

Question 1 - Solution

Use substitution. Let

$$u = \sqrt{1+x^2}.$$

Then

$$u^2 = 1 + x^2 \quad \Rightarrow \quad 2u \, du = 2x \, dx \quad \Rightarrow \quad x \, dx = u \, du.$$

Substitute into the integral:

$$\int \frac{x e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}} \, dx = \int \frac{u e^u}{u} \, du = \int e^u \, du.$$

Integrate:

$$\int e^u \, du = e^u.$$

Substitute back:

$$\boxed{e^{\sqrt{1+x^2}} + C}$$

Question 2

Evaluate the integral

$$\int \frac{\ln(x^2 + 1)}{x^2 + 1} x \, dx.$$

Question 2 - Solution

Use substitution. Let

$$u = \ln(x^2 + 1).$$

Then

$$du = \frac{2x}{x^2 + 1} dx \quad \Rightarrow \quad \frac{x}{x^2 + 1} dx = \frac{1}{2} du.$$

Substitute:

$$\int \frac{\ln(x^2 + 1)}{x^2 + 1} x dx = \frac{1}{2} \int u du.$$

Integrate:

$$\frac{1}{2} \int u du = \frac{1}{4} u^2.$$

Substitute back:

$$\boxed{\frac{1}{4} (\ln(x^2 + 1))^2 + C}$$

Question 3

Evaluate the integral

$$\int \frac{x^3}{\sqrt{1+x^4}} dx.$$

Question 3 - Solution

Use substitution. Let

$$u = 1 + x^4.$$

Then

$$du = 4x^3 dx \quad \Rightarrow \quad x^3 dx = \frac{1}{4} du.$$

Substitute into the integral:

$$\int \frac{x^3}{\sqrt{1+x^4}} dx = \frac{1}{4} \int u^{-1/2} du.$$

Integrate:

$$\frac{1}{4} \int u^{-1/2} du = \frac{1}{4} \cdot 2u^{1/2} = \frac{1}{2} \sqrt{u}.$$

Substitute back:

$$\boxed{\frac{1}{2} \sqrt{1+x^4} + C}$$

Question 4

Evaluate the integral

$$\int \frac{x \sin(x^2)}{\sqrt{1 + \cos(x^2)}} dx.$$

Question 4 - Solution

Use substitution. Let

$$u = \cos(x^2).$$

Then

$$du = -\sin(x^2) \cdot 2x \, dx \quad \Rightarrow \quad -\frac{1}{2} du = x \sin(x^2) \, dx.$$

Substitute into the integral:

$$\int \frac{x \sin(x^2)}{\sqrt{1 + \cos(x^2)}} \, dx = -\frac{1}{2} \int \frac{du}{\sqrt{1 + u}}.$$

Integrate:

$$-\frac{1}{2} \int (1 + u)^{-1/2} \, du = -\frac{1}{2} \cdot 2(1 + u)^{1/2} = -\sqrt{1 + u}.$$

Substitute back:

$$\boxed{-\sqrt{1 + \cos(x^2)} + C}$$

Question 5

Evaluate the integral

$$\int \frac{2x}{1 + \sqrt{1 + x^2}} dx.$$

Question 5 - Solution

Use substitution. Let

$$u = \sqrt{1 + x^2}.$$

Then

$$u^2 = 1 + x^2 \quad \Rightarrow \quad 2u \, du = 2x \, dx \quad \Rightarrow \quad 2x \, dx = 2u \, du.$$

Substitute into the integral:

$$\int \frac{2x}{1 + \sqrt{1 + x^2}} \, dx = \int \frac{2u}{1 + u} \, du.$$

Rewrite the integrand:

$$\frac{2u}{1 + u} = 2 \left(1 - \frac{1}{1 + u} \right).$$

Integrate:

$$\int 2 \left(1 - \frac{1}{1 + u} \right) \, du = 2u - 2 \ln(1 + u).$$

Substitute back:

$$\boxed{2\sqrt{1 + x^2} - 2 \ln(1 + \sqrt{1 + x^2}) + C}$$

Question 6

Evaluate the integral

$$\int \frac{x e^{1/(1+x^2)}}{(1+x^2)^2} dx.$$

Question 6 - Solution

Use substitution. Let

$$u = \frac{1}{1+x^2}.$$

Then

$$du = -\frac{2x}{(1+x^2)^2} dx \quad \Rightarrow \quad -\frac{1}{2} du = \frac{x}{(1+x^2)^2} dx.$$

Substitute into the integral:

$$\int \frac{x e^{1/(1+x^2)}}{(1+x^2)^2} dx = -\frac{1}{2} \int e^u du.$$

Integrate:

$$-\frac{1}{2} \int e^u du = -\frac{1}{2} e^u.$$

Substitute back:

$$\boxed{-\frac{1}{2} e^{1/(1+x^2)} + C}$$

Question 7

Evaluate

$$\int \frac{x}{\sqrt{1-x^2}} \ln(1 + \sqrt{1-x^2}) \, dx.$$

Question 7 - Solution

Let

$$u = \sqrt{1 - x^2}.$$

Then

$$du = \frac{-x}{\sqrt{1 - x^2}} dx \quad \Rightarrow \quad \frac{x}{\sqrt{1 - x^2}} dx = -du.$$

So the integral becomes

$$- \int \ln(1 + u) du.$$

Now integrate by parts. Take

$$w = \ln(1 + u), \quad dv = du,$$

so

$$dw = \frac{1}{1 + u} du, \quad v = u.$$

Then

$$- \int \ln(1 + u) du = - \left(u \ln(1 + u) - \int \frac{u}{1 + u} du \right).$$

Simplify

$$\frac{u}{1 + u} = 1 - \frac{1}{1 + u},$$

hence

$$\int \frac{u}{1 + u} du = \int \left(1 - \frac{1}{1 + u} \right) du = u - \ln(1 + u).$$

Therefore,

$$- (u \ln(1 + u) - u + \ln(1 + u)) = -u \ln(1 + u) + u - \ln(1 + u) + C.$$

Substitute back $u = \sqrt{1 - x^2}$:

$$\boxed{-\sqrt{1 - x^2} \ln(1 + \sqrt{1 - x^2}) + \sqrt{1 - x^2} - \ln(1 + \sqrt{1 - x^2}) + C}$$

Question 8

Evaluate the integral

$$\int \frac{3x^2}{1+x^3} \sqrt{\ln(1+x^3)} dx.$$

Question 8 - Solution

Use substitution. Let

$$u = \ln(1 + x^3).$$

Then

$$du = \frac{3x^2}{1 + x^3} dx.$$

Substitute into the integral:

$$\int \frac{3x^2}{1 + x^3} \sqrt{\ln(1 + x^3)} dx = \int \sqrt{u} du.$$

Integrate:

$$\int u^{1/2} du = \frac{2}{3} u^{3/2}.$$

Substitute back:

$$\boxed{\frac{2}{3} (\ln(1 + x^3))^{3/2} + C}$$

Question 9

Evaluate the integral

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx.$$

Question 9 - Solution

Rewrite the integrand to reveal a derivative pattern:

$$2 \cos(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}).$$

Let

$$u = \sqrt{x}.$$

Then

$$du = \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \quad dx = 2\sqrt{x} du.$$

Substitute into the integral:

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx = \int \frac{\cos(u)}{\sqrt{x}} \cdot 2\sqrt{x} du = 2 \int \cos u du.$$

Integrate:

$$2 \int \cos u du = 2 \sin u.$$

Substitute back:

$$\boxed{2 \sin(\sqrt{x}) + C}$$

Question 10

Evaluate the integral

$$\int x \ln(1 - x^2) dx.$$

Question 10 - Solution

Let

$$u = 1 - x^2.$$

Then

$$du = -2x \, dx.$$

Rewrite the integral:

$$\int x \ln(1 - x^2) \, dx = -\frac{1}{2} \int \ln u \, du.$$

Now integrate:

$$\int \ln u \, du = u \ln u - u.$$

Therefore,

$$-\frac{1}{2}(u \ln u - u) + C.$$

Substitute back:

$$-\frac{1}{2}[(1 - x^2) \ln(1 - x^2) - (1 - x^2)] + C$$