

Question 1

Compute the indefinite integral

$$\int x^2 e^x dx.$$

Question 1 - Solution

Use integration by parts with

$$u = x^2, \quad dv = e^x dx.$$

Then

$$du = 2x dx, \quad v = e^x.$$

So

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx.$$

Apply integration by parts again to $\int 2x e^x dx$:

$$u = 2x, \quad dv = e^x dx \Rightarrow du = 2 dx, \quad v = e^x.$$

Thus,

$$\int 2x e^x dx = 2x e^x - 2 \int e^x dx = 2x e^x - 2e^x.$$

Hence,

$$\int x^2 e^x dx = e^x (x^2 - 2x + 2) + C.$$

Final Answer:

$$\int x^2 e^x dx = e^x (x^2 - 2x + 2) + C$$

Question 2

Evaluate the integral

$$\int x^2 \ln(x^2 + 1) dx.$$

Question 2 - Solution

Use integration by parts. Let

$$u = \ln(x^2 + 1), \quad dv = x^2 dx.$$

Then

$$du = \frac{2x}{x^2 + 1} dx, \quad v = \frac{x^3}{3}.$$

Apply integration by parts:

$$\int x^2 \ln(x^2 + 1) dx = \frac{x^3}{3} \ln(x^2 + 1) - \frac{2}{3} \int \frac{x^4}{x^2 + 1} dx.$$

Simplify the integrand by division:

$$\frac{x^4}{x^2 + 1} = x^2 - 1 + \frac{1}{x^2 + 1}.$$

Thus,

$$\int \frac{x^4}{x^2 + 1} dx = \int (x^2 - 1) dx + \int \frac{1}{x^2 + 1} dx = \frac{x^3}{3} - x + \arctan x.$$

Substitute back:

$$\int x^2 \ln(x^2 + 1) dx = \frac{x^3}{3} \ln(x^2 + 1) - \frac{2}{3} \left(\frac{x^3}{3} - x + \arctan x \right).$$

Final Answer:

$$\frac{x^3}{3} \ln(x^2 + 1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2}{3} \arctan x + C$$

Question 3

Evaluate the integral

$$\int \frac{x^3}{(x^2 + 1)^2} dx.$$

Question 3 - Solution

Rewrite the integrand by separating powers of x :

$$\frac{x^3}{(x^2 + 1)^2} = \frac{x(x^2 + 1 - 1)}{(x^2 + 1)^2} = \frac{x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2}.$$

Split the integral:

$$\int \frac{x}{x^2 + 1} dx - \int \frac{x}{(x^2 + 1)^2} dx.$$

For the first integral, let $u = x^2 + 1$, $du = 2x dx$:

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln(x^2 + 1).$$

For the second integral, use the same substitution:

$$\int \frac{x}{(x^2 + 1)^2} dx = \frac{1}{2} \int u^{-2} du = -\frac{1}{2u} = -\frac{1}{2(x^2 + 1)}.$$

Combine results:

$$\boxed{\frac{1}{2} \ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} + C}$$

Question 4

Evaluate the integral

$$\int x \sqrt{x^2 + 4} \ln(x^2 + 4) dx.$$

Question 4 - Solution

Use substitution. Let

$$u = x^2 + 4, \quad du = 2x \, dx.$$

Then

$$\int x\sqrt{x^2 + 4} \ln(x^2 + 4) \, dx = \frac{1}{2} \int u^{1/2} \ln u \, du.$$

Apply integration by parts with

$$w = \ln u, \quad dv = u^{1/2} \, du.$$

Then

$$dw = \frac{1}{u} \, du, \quad v = \frac{2}{3} u^{3/2}.$$

Thus,

$$\int u^{1/2} \ln u \, du = \frac{2}{3} u^{3/2} \ln u - \frac{2}{3} \int u^{1/2} \, du = \frac{2}{3} u^{3/2} \ln u - \frac{4}{9} u^{3/2}.$$

Multiply by $\frac{1}{2}$ and substitute back:

$$\frac{1}{3} (x^2 + 4)^{3/2} \ln(x^2 + 4) - \frac{2}{9} (x^2 + 4)^{3/2} + C$$

Question 5

Evaluate

$$\int \frac{x^2}{\sqrt{x^2 + 1}} dx.$$

Question 5 - Solution**Step 1: Rewrite the integrand**

Write $x^2 = (x^2 + 1) - 1$:

$$\frac{x^2}{\sqrt{x^2 + 1}} = \frac{(x^2 + 1) - 1}{\sqrt{x^2 + 1}} = \sqrt{x^2 + 1} - \frac{1}{\sqrt{x^2 + 1}}.$$

Thus,

$$\int \frac{x^2}{\sqrt{x^2 + 1}} dx = \int \sqrt{x^2 + 1} dx - \int \frac{1}{\sqrt{x^2 + 1}} dx.$$

Step 2: Trigonometric substitution

Let

$$x = \tan \theta, \quad dx = \sec^2 \theta d\theta.$$

Then

$$\sqrt{x^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta.$$

Step 3: Evaluate $\int \sqrt{x^2 + 1} dx$

Substitute:

$$\int \sqrt{x^2 + 1} dx = \int \sec \theta (\sec^2 \theta d\theta) = \int \sec^3 \theta d\theta.$$

To evaluate $\int \sec^3 \theta d\theta$, write

$$\sec^3 \theta = \sec \theta \sec^2 \theta.$$

Let

$$u = \sec \theta, \quad dv = \sec^2 \theta d\theta.$$

Then

$$du = \sec \theta \tan \theta d\theta, \quad v = \tan \theta.$$

By integration by parts,

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \tan \theta (\sec \theta \tan \theta) d\theta.$$

Simplify the integrand:

$$\tan^2 \theta = \sec^2 \theta - 1,$$

so

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta.$$

Distribute:

$$= \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta.$$

Move the repeated integral to the left:

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta.$$

Now compute $\int \sec \theta d\theta$:

$$\int \sec \theta d\theta = \int \frac{\sec \theta (\sec \theta + \tan \theta)}{\sec \theta + \tan \theta} d\theta.$$

Let

$$u = \sec \theta + \tan \theta, \quad du = (\sec \theta \tan \theta + \sec^2 \theta) d\theta = \sec \theta (\sec \theta + \tan \theta) d\theta.$$

Hence,

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C.$$

Substitute back:

$$2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|.$$

Therefore,

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C.$$

Returning to x :

$$\sec \theta = \sqrt{x^2 + 1}, \quad \tan \theta = x.$$

Thus,

$$\int \sqrt{x^2 + 1} \, dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \ln(x + \sqrt{x^2 + 1}) \right) + C.$$

Step 4: Evaluate $\int \frac{1}{\sqrt{x^2 + 1}} \, dx$

Substitute again:

$$\int \frac{1}{\sqrt{x^2 + 1}} \, dx = \int \frac{1}{\sec \theta} (\sec^2 \theta \, d\theta) = \int \sec \theta \, d\theta.$$

From above,

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln(x + \sqrt{x^2 + 1}) + C.$$

Step 5: Combine results

$$\int \frac{x^2}{\sqrt{x^2 + 1}} \, dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \ln(x + \sqrt{x^2 + 1}) \right) - \ln(x + \sqrt{x^2 + 1}) + C.$$

Combine logarithms:

$$= \frac{x}{2} \sqrt{x^2 + 1} - \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + C.$$

Final Answer:

$$\boxed{\int \frac{x^2}{\sqrt{x^2 + 1}} \, dx = \frac{x}{2} \sqrt{x^2 + 1} - \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + C}$$

Question 6

Evaluate the integral

$$\int \frac{x^3 \ln x}{1+x^2} dx.$$

Question 6 - Solution

Rewrite the integrand by splitting the rational expression:

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}.$$

Thus,

$$\int \frac{x^3 \ln x}{1+x^2} dx = \int x \ln x dx - \int \frac{x \ln x}{1+x^2} dx.$$

First integral: Use integration by parts with

$$u = \ln x, \quad dv = x dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{x^2}{2}.$$

So

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}.$$

Second integral: Use integration by parts with

$$u = \ln x, \quad dv = \frac{x}{1+x^2} dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{1}{2} \ln(1+x^2).$$

Hence,

$$\int \frac{x \ln x}{1+x^2} dx = \frac{1}{2} \ln x \ln(1+x^2) - \frac{1}{2} \int \frac{\ln(1+x^2)}{x} dx.$$

The remaining integral is non-elementary and is expressed using the dilogarithm function:

$$\int \frac{\ln(1+x^2)}{x} dx = -\frac{1}{2} \text{Li}_2(-x^2).$$

Therefore,

$$\int \frac{x \ln x}{1+x^2} dx = \frac{1}{2} \ln x \ln(1+x^2) + \frac{1}{4} \text{Li}_2(-x^2).$$

Final Answer:

$$\boxed{\frac{x^2}{2} \ln x - \frac{x^2}{4} - \frac{1}{2} \ln x \ln(1+x^2) - \frac{1}{4} \text{Li}_2(-x^2) + C}$$

Question 7

Evaluate the integral

$$\int \frac{\sqrt{x^2 + 1}}{x} dx.$$

Question 7 - Solution

For $x \neq 0$, write $s = \sqrt{1+x^2}$, so $ds = (x/s) dx$ and $x^2 = s^2 - 1$.

$$\int \frac{\sqrt{1+x^2}}{x} dx = \int \frac{s^2}{s^2-1} ds = s + \frac{1}{2} \ln \left| \frac{s-1}{s+1} \right| + C.$$

Since $(s-1)(s+1) = x^2$, this becomes

$$\boxed{\sqrt{1+x^2} + \ln \left| \frac{x}{1+\sqrt{1+x^2}} \right| + C.}$$

The formula is valid on either interval $(-\infty, 0)$ or $(0, \infty)$; constants may differ between intervals.

Question 8

Evaluate the integral

$$\int (x^2 + 1) \ln(x^2 + 1) dx.$$

Question 8 - Solution

Using substitution does not work!

$$u = x^2 + 1, \quad du = 2x \, dx.$$

Rewrite the integrand:

$$(x^2 + 1) \ln(x^2 + 1) \, dx = u \ln u \cdot \frac{du}{2x}.$$

Instead, apply integration by parts directly.

Let

$$u = \ln(x^2 + 1), \quad dv = (x^2 + 1) \, dx.$$

Then

$$du = \frac{2x}{x^2 + 1} \, dx, \quad v = \frac{x^3}{3} + x.$$

Apply integration by parts:

$$\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \int \left(\frac{x^3}{3} + x \right) \frac{2x}{x^2 + 1} \, dx.$$

Simplify the remaining integral:

$$\left(\frac{x^3}{3} + x \right) \frac{2x}{x^2 + 1} = \frac{2x^4}{3(x^2 + 1)} + \frac{2x^2}{x^2 + 1}.$$

Use algebraic division:

$$\frac{x^4}{x^2 + 1} = x^2 - 1 + \frac{1}{x^2 + 1}.$$

Thus,

$$\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \int \left(\frac{2}{3}(x^2 - 1) + \frac{2}{3(x^2 + 1)} + \frac{2x^2}{x^2 + 1} \right) \, dx.$$

Compute term by term:

$$\begin{aligned} \int (x^2 - 1) \, dx &= \frac{x^3}{3} - x, \\ \int \frac{1}{x^2 + 1} \, dx &= \arctan x, \quad \int \frac{x^2}{x^2 + 1} \, dx = x - \arctan x. \end{aligned}$$

Putting everything together,

$$\boxed{\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \frac{2x^3}{9} - \frac{4x}{3} + \frac{4}{3} \arctan x + C}$$

Question 9

Evaluate the integral

$$\int \frac{x^4}{(1+x^2)^{3/2}} dx.$$

Question 9 - Solution

Use the trigonometric substitution

$$x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta.$$

Then

$$1 + x^2 = \sec^2 \theta, \quad (1 + x^2)^{3/2} = \sec^3 \theta.$$

Substitute into the integral:

$$\int \frac{\tan^4 \theta}{\sec^3 \theta} \sec^2 \theta \, d\theta = \int \tan^4 \theta \cos \theta \, d\theta.$$

Rewrite $\tan^4 \theta$:

$$\tan^4 \theta = (\sec^2 \theta - 1)^2 = \sec^4 \theta - 2\sec^2 \theta + 1.$$

Thus,

$$\int (\sec^3 \theta - 2\sec \theta + \cos \theta) \, d\theta.$$

Integrate term by term:

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln(\sec \theta + \tan \theta),$$

$$\int \sec \theta \, d\theta = \ln(\sec \theta + \tan \theta), \quad \int \cos \theta \, d\theta = \sin \theta.$$

Combining,

$$\frac{1}{2} \sec \theta \tan \theta - \frac{3}{2} \ln(\sec \theta + \tan \theta) + \sin \theta + C.$$

Substitute back:

$$\sec \theta = \sqrt{1 + x^2}, \quad \tan \theta = x, \quad \sin \theta = \frac{x}{\sqrt{1 + x^2}}.$$

Therefore,

$$\boxed{\frac{x}{2} \sqrt{1 + x^2} - \frac{3}{2} \ln(x + \sqrt{1 + x^2}) + \frac{x}{\sqrt{1 + x^2}} + C}$$

Question 10

Evaluate the indefinite integral:

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx.$$

Question 10 - Solution

We are given:

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx$$

Rewrite the numerator:

$$x^2 + 2x + 2 = (x^2 + 2x + 5) - 3.$$

Then

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx = \int \sqrt{x^2 + 2x + 5} dx - 3 \int \frac{dx}{\sqrt{x^2 + 2x + 5}}.$$

Complete the square:

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Let

$$t = x + 1.$$

Then

$$x^2 + 2x + 5 = t^2 + 4$$

and

$$dt = dx.$$

So the integral becomes:

$$\int \sqrt{t^2 + 4} dt - 3 \int \frac{dt}{\sqrt{t^2 + 4}}.$$

Using the formulas

$$\int \sqrt{t^2 + a^2} dt = \frac{t}{2} \sqrt{t^2 + a^2} + \frac{a^2}{2} \ln |t + \sqrt{t^2 + a^2}|$$

and

$$\int \frac{dt}{\sqrt{t^2 + a^2}} = \ln |t + \sqrt{t^2 + a^2}|,$$

with $a = 2$, we get:

$$\int \sqrt{t^2 + 4} dt = \frac{t}{2} \sqrt{t^2 + 4} + 2 \ln |t + \sqrt{t^2 + 4}|$$

and

$$3 \int \frac{dt}{\sqrt{t^2 + 4}} = 3 \ln |t + \sqrt{t^2 + 4}|.$$

Therefore,

$$\int \frac{x^2 + 2x + 2}{\sqrt{x^2 + 2x + 5}} dx = \frac{t}{2} \sqrt{t^2 + 4} + 2 \ln |t + \sqrt{t^2 + 4}| - 3 \ln |t + \sqrt{t^2 + 4}|.$$

Simplify:

$$= \frac{t}{2} \sqrt{t^2 + 4} - \ln |t + \sqrt{t^2 + 4}| + C.$$

Substitute back $t = x + 1$:

$$\boxed{\frac{x+1}{2} \sqrt{x^2 + 2x + 5} - \ln |x + 1 + \sqrt{x^2 + 2x + 5}| + C}$$