

Question 1

Evaluate the indefinite integral

$$\int \left(x^2 e^{x^3} + \frac{1}{x} \sqrt{1 + \ln x} - \frac{3x^2}{1 + x^3} \right) dx.$$

Question 1 - Solution

We integrate term by term.

First term:

$$\int x^2 e^{x^3} dx.$$

Let $u = x^3$, so $du = 3x^2 dx$:

$$\int x^2 e^{x^3} dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^{x^3}.$$

Second term:

$$\int \frac{1}{x} \sqrt{1 + \ln x} dx.$$

Let $u = 1 + \ln x$, so $du = \frac{1}{x} dx$:

$$\int \sqrt{u} du = \frac{2}{3} u^{3/2} = \frac{2}{3} (1 + \ln x)^{3/2}.$$

Third term:

$$\int \frac{3x^2}{1 + x^3} dx.$$

Let $u = 1 + x^3$, so $du = 3x^2 dx$:

$$\int \frac{3x^2}{1 + x^3} dx = \int \frac{du}{u} = \ln(1 + x^3).$$

Final Answer:

$\frac{1}{3} e^{x^3} + \frac{2}{3} (1 + \ln x)^{3/2} - \ln(1 + x^3) + C$
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Question 2

Find the most general antiderivative of

$$\int \left(e^{2x} \sin x + \frac{1}{(1-x)^2} \right) dx.$$

Question 2 - Solution

We evaluate each term separately.

First term:

$$\int e^{2x} \sin x \, dx.$$

Use integration by parts. Let

$$u = \sin x, \quad dv = e^{2x} \, dx.$$

Then

$$du = \cos x \, dx, \quad v = \frac{1}{2}e^{2x}.$$

So,

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = \cos x, \quad dv = e^{2x} \, dx.$$

Then

$$du = -\sin x \, dx, \quad v = \frac{1}{2}e^{2x}.$$

Thus,

$$\int e^{2x} \cos x \, dx = \frac{1}{2}e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x \, dx.$$

Substitute this back:

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x \, dx.$$

Add $\frac{1}{4} \int e^{2x} \sin x \, dx$ to both sides:

$$\frac{5}{4} \int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x.$$

Solve:

$$\int e^{2x} \sin x \, dx = \frac{e^{2x}}{5} (2 \sin x - \cos x).$$

Second term:

$$\int \frac{1}{(1-x)^2} \, dx.$$

Let

$$u = 1 - x, \quad du = -dx.$$

Then

$$\int \frac{1}{(1-x)^2} \, dx = - \int u^{-2} \, du = u^{-1} = \frac{1}{1-x}.$$

Final Answer:

$$\boxed{\frac{e^{2x}}{5} (2 \sin x - \cos x) + \frac{1}{1-x} + C}$$

Question 3

Determine an antiderivative of

$$\int \frac{x^3}{\sqrt{1-x^2}} dx.$$

Question 3 - Solution

For $|x| < 1$, set $u = 1 - x^2$, so $du = -2x \, dx$ and $x^2 = 1 - u$.

$$\int \frac{x^3}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int (1-u)u^{-1/2} \, du = -\sqrt{u} + \frac{1}{3}u^{3/2} + C.$$

$$-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.$$

Differentiation gives $x/\sqrt{1-x^2} - x\sqrt{1-x^2} = x^3/\sqrt{1-x^2}$.

Question 4

Evaluate the integral

$$\int \left(\sqrt{x} \ln x + \frac{1}{\sqrt{x}} \right) dx.$$

Question 4 - Solution

We first rewrite the integrand using exponents:

$$\sqrt{x} \ln x + \frac{1}{\sqrt{x}} = x^{1/2} \ln x + x^{-1/2}.$$

For the first term, apply integration by parts with

$$u = \ln x, \quad dv = x^{1/2} dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{2}{3} x^{3/2}.$$

Thus,

$$\int x^{1/2} \ln x dx = \frac{2}{3} x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} dx = \frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2}.$$

For the second term,

$$\int x^{-1/2} dx = 2x^{1/2}.$$

Combining results:

$$\boxed{\frac{2}{3} x^{3/2} \ln x - \frac{4}{9} x^{3/2} + 2x^{1/2} + C}$$

Question 5

Find an antiderivative of

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx.$$

Question 5 - Solution

Split the integral:

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx = \int \frac{\arctan x}{x} dx + \int \frac{x^2}{1+x^2} dx.$$

1) The rational part

Rewrite:

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

So

$$\int \frac{x^2}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2} \right) dx = x - \arctan x.$$

2) The non-elementary part $\int \frac{\arctan x}{x} dx$

A common attempt is integration by parts, which produces

$$\int \frac{\arctan x}{x} dx = \arctan x \ln|x| - \int \frac{\ln|x|}{1+x^2} dx,$$

and then the substitution $x = \tan \theta$ turns the remaining integral into

$$\int \frac{\ln|x|}{1+x^2} dx = \int \ln(\tan \theta) d\theta.$$

The key clarification is:

$\int \ln(\tan \theta) d\theta$ **is not an elementary antiderivative.** So any step claiming

$$\int \ln(\tan \theta) d\theta = \theta \ln(\tan \theta) - \tan \theta + C$$

is incorrect (differentiate it to check).

A correct closed form uses the dilogarithm function Li_2 . Start from

$$\arctan x = \frac{1}{2i} (\ln(1+ix) - \ln(1-ix)).$$

Then

$$\int \frac{\arctan x}{x} dx = \frac{1}{2i} \int \frac{\ln(1+ix) - \ln(1-ix)}{x} dx.$$

Use the identity

$$\frac{d}{dx} \text{Li}_2(u) = -\ln(1-u) \frac{u'}{u},$$

so, for real x ,

$$\frac{d}{dx} \text{Li}_2(-ix) = -\frac{\ln(1+ix)}{x}, \quad \frac{d}{dx} \text{Li}_2(ix) = -\frac{\ln(1-ix)}{x}.$$

Therefore,

$$\int \frac{\ln(1+ix)}{x} dx = -\text{Li}_2(-ix) + C, \quad \int \frac{\ln(1-ix)}{x} dx = -\text{Li}_2(ix) + C,$$

and hence

$$\int \frac{\arctan x}{x} dx = \frac{1}{2i} (\text{Li}_2(ix) - \text{Li}_2(-ix)) + C.$$

Final Answer

Combine both parts:

$$\int \left(\frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx = (x - \arctan x) + \frac{1}{2i} \left(\operatorname{Li}_2(ix) - \operatorname{Li}_2(-ix) \right) + C.$$

Question 6

Find the general antiderivative of

$$\int \left(x e^{-x^2} + \ln(1+x) \right) dx.$$

Question 6 - Solution

We evaluate each term separately.

First term:

$$\int x e^{-x^2} dx.$$

Let $u = -x^2$, so $du = -2x dx$:

$$\int x e^{-x^2} dx = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^{-x^2}.$$

Second term:

$$\int \ln(1+x) dx.$$

Use integration by parts with

$$u = \ln(1+x), \quad dv = dx.$$

Then

$$du = \frac{1}{1+x} dx, \quad v = x.$$

Thus,

$$\int \ln(1+x) dx = x \ln(1+x) - \int \frac{x}{1+x} dx.$$

Rewrite the integrand:

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

Hence,

$$\int \frac{x}{1+x} dx = x - \ln(1+x).$$

Combining results:

$$-\frac{1}{2} e^{-x^2} + x \ln(1+x) - x + \ln(1+x) + C$$

Question 7

Find an antiderivative of

$$\int \left(\frac{\sqrt{1+2x}}{x} - \frac{1}{x\sqrt{1+2x}} \right) dx.$$

Question 7 - Solution

Combine the terms into a single fraction:

$$\frac{\sqrt{1+2x}}{x} - \frac{1}{x\sqrt{1+2x}} = \frac{(1+2x) - 1}{x\sqrt{1+2x}} = \frac{2x}{x\sqrt{1+2x}} = \frac{2}{\sqrt{1+2x}}.$$

Thus the integral simplifies to

$$\int \frac{2}{\sqrt{1+2x}} dx.$$

Let

$$u = 1 + 2x, \quad du = 2 dx.$$

Then

$$\int \frac{2}{\sqrt{1+2x}} dx = \int u^{-1/2} du = 2u^{1/2}.$$

Substitute back:

$$\boxed{2\sqrt{1+2x} + C}$$

Question 8

Find the general antiderivative of

$$\int \left(\ln(x + \sqrt{1 + x^2}) + \frac{x^2}{\sqrt{1 + x^2}} \right) dx.$$

Question 8 - Solution

Split the integral:

$$\int \ln(x + \sqrt{1+x^2}) \, dx + \int \frac{x^2}{\sqrt{1+x^2}} \, dx.$$

Part A: $\int \ln(x + \sqrt{1+x^2}) \, dx$

Use integration by parts with

$$u = \ln(x + \sqrt{1+x^2}), \quad dv = dx, \quad v = x.$$

Let

$$g(x) = x + \sqrt{1+x^2}.$$

Then

$$du = \frac{g'(x)}{g(x)} \, dx, \quad g'(x) = 1 + \frac{x}{\sqrt{1+x^2}}.$$

Thus

$$du = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \, dx = \frac{1}{\sqrt{1+x^2}} \, dx.$$

Therefore,

$$\int \ln(x + \sqrt{1+x^2}) \, dx = x \ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} \, dx.$$

Let $w = 1 + x^2$, $dw = 2x \, dx$:

$$\int \frac{x}{\sqrt{1+x^2}} \, dx = \frac{1}{2} \int w^{-1/2} \, dw = \sqrt{1+x^2}.$$

Hence,

$$\int \ln(x + \sqrt{1+x^2}) \, dx = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}.$$

Part B: $\int \frac{x^2}{\sqrt{1+x^2}} \, dx$ (use an easier trigonometric substitution)

Let

$$x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta, \quad \sqrt{1+x^2} = \sec \theta.$$

Then

$$\int \frac{x^2}{\sqrt{1+x^2}} \, dx = \int \frac{\tan^2 \theta}{\sec \theta} \sec^2 \theta \, d\theta = \int \tan^2 \theta \sec \theta \, d\theta.$$

Use $\tan^2 \theta = \sec^2 \theta - 1$:

$$\int \tan^2 \theta \sec \theta \, d\theta = \int (\sec^3 \theta - \sec \theta) \, d\theta.$$

Evaluate each term directly:

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta|,$$

and for $\int \sec^3 \theta \, d\theta$, integrate by parts:

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).$$

Thus,

$$\int (\sec^3 \theta - \sec \theta) \, d\theta = \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta|.$$

Return to x using $\tan \theta = x$, $\sec \theta = \sqrt{1+x^2}$:

$$\int \frac{x^2}{\sqrt{1+x^2}} dx = \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right).$$

Combine Parts A and B:

$$x \ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C$$

Question 9

Find the general antiderivative of

$$\int \frac{\arctan x}{x^2} dx.$$

Question 9 - Solution

We use integration by parts.

Let

$$u = \arctan x, \quad dv = \frac{1}{x^2} dx.$$

Then

$$du = \frac{1}{1+x^2} dx, \quad v = -\frac{1}{x}.$$

Applying integration by parts,

$$\int \frac{\arctan x}{x^2} dx = -\frac{\arctan x}{x} + \int \frac{1}{x(1+x^2)} dx.$$

Now decompose the integrand:

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}.$$

Thus,

$$\int \frac{1}{x(1+x^2)} dx = \ln|x| - \frac{1}{2} \ln(1+x^2).$$

Final Answer:

$-\frac{\arctan x}{x} + \ln x - \frac{1}{2} \ln(1+x^2) + C$
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Question 10

Find an antiderivative of

$$\int \frac{x \ln(1+x^2)}{1+x^2} dx.$$

Question 10 - Solution

Use the substitution

$$u = \ln(1 + x^2).$$

Then

$$du = \frac{2x}{1+x^2} dx, \quad \frac{1}{2} du = \frac{x}{1+x^2} dx.$$

Rewrite the integral:

$$\int \frac{x \ln(1+x^2)}{1+x^2} dx = \frac{1}{2} \int u du.$$

Integrate:

$$\frac{1}{2} \int u du = \frac{1}{4} u^2.$$

Substitute back:

$$\boxed{\frac{1}{4} (\ln(1+x^2))^2 + C}$$