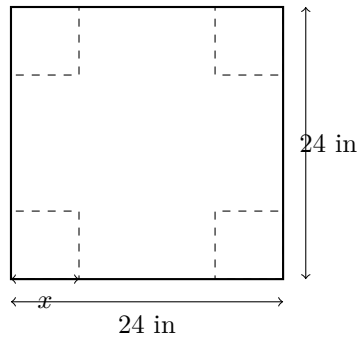


Question 1

A square piece of cardboard 24 inches on each side is to be made into an open-top box by cutting equal squares from each corner and folding up the sides. What size square should be cut from each corner to maximize the volume of the box?

Diagram:



Question 1 - Solution

Let x be the length of the square cut from each corner.

After folding, the dimensions of the box will be:

Height = x

Length and width = $24 - 2x$

Volume:

$$V(x) = x(24 - 2x)^2$$

Step 1: Expand volume function

$$V(x) = x(576 - 96x + 4x^2) = 576x - 96x^2 + 4x^3$$

Step 2: Find critical points

$$V'(x) = 576 - 192x + 12x^2 \Rightarrow 12x^2 - 192x + 576 = 0 \Rightarrow x^2 - 16x + 48 = 0 \Rightarrow x = 8 \pm \sqrt{16} \Rightarrow x = 8 \pm 4$$

$$x = 4 \quad \text{or} \quad x = 12$$

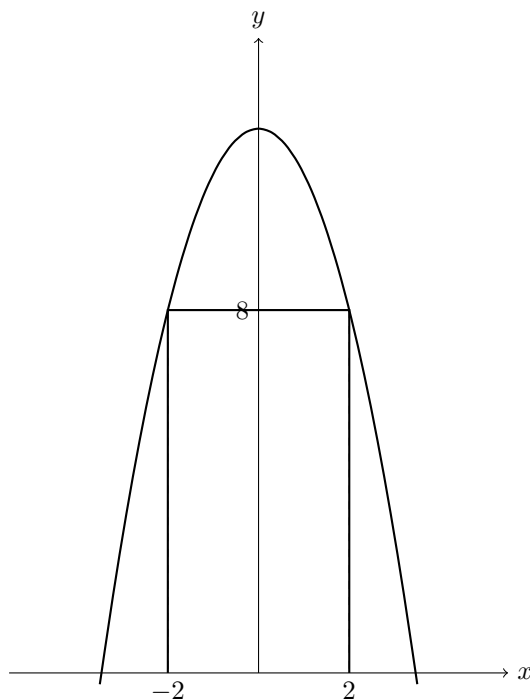
But $x = 12$ makes $24 - 2x = 0 \rightarrow$ not valid

Valid critical point: $x = 4$

Answer: Cut out squares of size 4 in to maximize volume

Question 2

A rectangle is inscribed under the parabola $y = 12 - x^2$ such that its base lies on the x -axis and its upper corners lie on the parabola.



Find the dimensions of the rectangle of maximum area.

Question 2 - Solution

Let the top corners of the rectangle be (x, y) and $(-x, y)$, where the rectangle is symmetric about the y -axis.

From the parabola:

$$y = 12 - x^2$$

The width of the rectangle is $2x$, and the height is $y = 12 - x^2$

Area Function:

$$A(x) = \text{width} \times \text{height} = 2x(12 - x^2) = 24x - 2x^3$$

Differentiate:

$$A'(x) = 24 - 6x^2$$

Set derivative to zero:

$$24 - 6x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

Check Maximum:

Second derivative:

$$A''(x) = -12x \Rightarrow A''(2) = -24 < 0$$

So, $x = 2$ gives a maximum.

Final Dimensions:

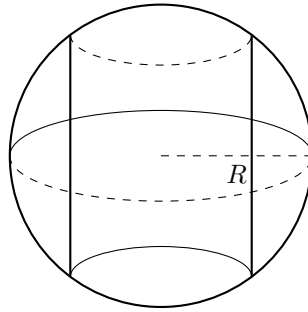
$$x = 2 \Rightarrow \text{width} = 2x = 4$$

$$y = 12 - x^2 = 12 - 4 = 8$$

Width = 4
Height = 8
Maximum Area = 32

Question 3

A right circular cylinder is inscribed inside a sphere of radius R .



Find the dimensions (radius and height) of the cylinder that ****maximize its volume****, and express the maximum volume in terms of R .

Question 3 - Solution

Let the radius of the cylinder be r and height h . Since the cylinder fits in the sphere of radius R , we use the Pythagorean relation from the triangle formed by r , $h/2$, and R :

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

Volume of the Cylinder:

$$V = \pi r^2 h = \pi r^2 \cdot 2\sqrt{R^2 - r^2} = 2\pi r^2 \sqrt{R^2 - r^2}$$

Maximize Volume:

Let:

$$V(r) = 2\pi r^2 \sqrt{R^2 - r^2}$$

Take the derivative (using product and chain rule):

$$\begin{aligned} V'(r) &= 2\pi \left[2r\sqrt{R^2 - r^2} + r^2 \cdot \frac{1}{2}(R^2 - r^2)^{-1/2}(-2r) \right] \\ &= 2\pi \left[2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}} \right] \end{aligned}$$

Set $V'(r) = 0$:

$$2r\sqrt{R^2 - r^2} = \frac{r^3}{\sqrt{R^2 - r^2}} \Rightarrow 2(R^2 - r^2) = r^2 \Rightarrow 2R^2 - 2r^2 = r^2 \Rightarrow 3r^2 = 2R^2 \Rightarrow r^2 = \frac{2}{3}R^2$$

$$r = R\sqrt{\frac{2}{3}}, \quad h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \frac{2}{3}R^2} = 2\sqrt{\frac{1}{3}R^2} = 2R\sqrt{\frac{1}{3}}$$

Maximum Volume:

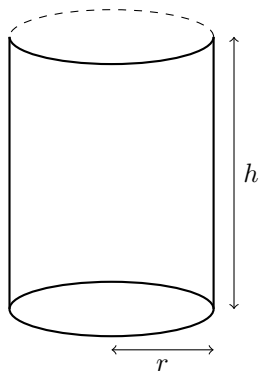
$$V = \pi r^2 h = \pi \cdot \frac{2}{3}R^2 \cdot 2R\sqrt{\frac{1}{3}} = \frac{4\pi R^3}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

$$\boxed{\begin{aligned} r &= R\sqrt{\frac{2}{3}} \\ h &= 2R\sqrt{\frac{1}{3}} \\ V_{\max} &= \frac{4\pi R^3}{3\sqrt{3}} \end{aligned}}$$

Question 4

A closed right circular cylinder has a fixed volume of $V = 1000 \text{ cm}^3$.

Find the radius and height of the cylinder that minimizes its surface area. Also determine the minimum surface area.



Question 4 - Solution

Let the radius be r , and the height be h . We are given:

$$V = \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$$

Surface Area of the Cylinder:

$$A = 2\pi r^2 + 2\pi r h \Rightarrow A(r) = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

Minimize $A(r)$:

Take derivative:

$$A'(r) = 4\pi r - \frac{2000}{r^2}$$

Set $A'(r) = 0$:

$$4\pi r = \frac{2000}{r^2} \Rightarrow 4\pi r^3 = 2000 \Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

Now compute h :

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi} \cdot \left(\frac{\pi}{500}\right)^{2/3} = 1000 \cdot \pi^{-1/3} \cdot 500^{-2/3}$$

Minimum Surface Area:

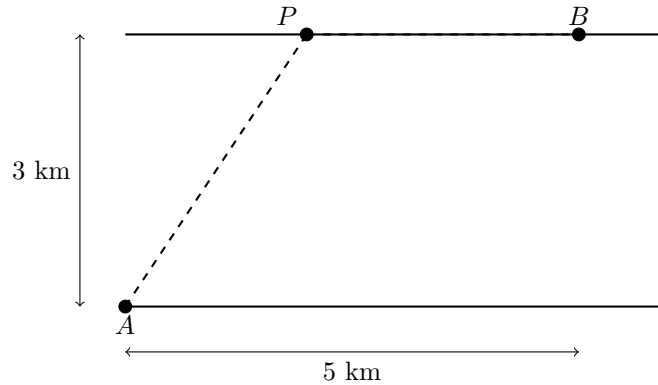
$$A = 2\pi r^2 + \frac{2000}{r} = 2\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2 + \frac{2000}{\sqrt[3]{\frac{500}{\pi}}}$$

$r = \sqrt[3]{\frac{500}{\pi}}$
$h = \frac{1000}{\pi r^2}$
$A_{\min} = 2\pi r^2 + \frac{2000}{r}$

Question 5

A person is standing at point A on the shore of a straight river that is 3 km wide. Point B lies on the opposite shore, 5 km downstream from the point directly opposite A . The person can run at 10 km/h along the shore and swim at 5 km/h directly across the river or at an angle.

The person first swims from A to P , then runs along the opposite shore to B . At what point P between the point directly opposite A and B should the person land to minimize total time to reach B ?



Question 5 - Solution

Let the swimmer land at point P , which is x km from the point directly across from A .

Step 1: Write distances:

Swim distance: $\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}$, Run distance: $5 - x$

Step 2: Write time expression:

Swimming speed: 5 km/h , Running speed: 10 km/h

$$T(x) = \frac{\sqrt{x^2 + 9}}{5} + \frac{5 - x}{10}$$

Step 3: Minimize time — take derivative:

$$T'(x) = \frac{x}{5\sqrt{x^2 + 9}} - \frac{1}{10}$$

Set $T'(x) = 0$:

$$\frac{x}{5\sqrt{x^2 + 9}} = \frac{1}{10} \Rightarrow \frac{2x}{\sqrt{x^2 + 9}} = 1 \Rightarrow 4x^2 = x^2 + 9 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}$$

Step 4: Answer

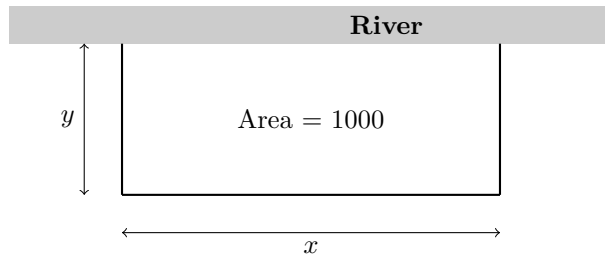
$$x = \sqrt{3} \approx 1.732 \text{ km}$$

$T_{\min}$ = when the person lands $\sqrt{3}$ km from the point directly across from A .

Question 6

A rectangular enclosure is to be built adjacent to a river (no fence is needed along the river side). The other three sides are to be fenced. The area of the enclosure must be 1000 square meters.

If the fencing along the two sides perpendicular to the river costs \$10 per meter and the fencing along the side parallel to the river costs \$5 per meter, what dimensions will minimize the total cost?



Question 6 - Solution

Let: - x = length along the river (no fence needed) - y = depth away from river (fence on both sides)

Area constraint:

$$xy = 1000 \Rightarrow y = \frac{1000}{x}$$

Cost function: - Two sides of length y at \$10/m: cost = $2(10)(y) = 20y$ - One side of length x at \$5/m: cost = $5x$

$$C(x) = 5x + 20 \cdot \frac{1000}{x} = 5x + \frac{20000}{x}$$

Minimize cost — take derivative:

$$C'(x) = 5 - \frac{20000}{x^2}$$

Set derivative to zero:

$$5 = \frac{20000}{x^2} \Rightarrow x^2 = 4000 \Rightarrow x = \sqrt{4000} = 20\sqrt{10}$$

Find y :

$$y = \frac{1000}{x} = \frac{1000}{20\sqrt{10}} = \frac{50}{\sqrt{10}} = 5\sqrt{10}$$

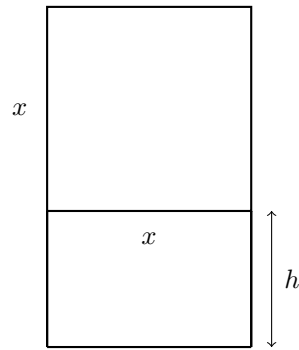
Answer:

$x = 20\sqrt{10} \approx 63.25 \text{ m}$ $y = 5\sqrt{10} \approx 15.81 \text{ m}$

These dimensions minimize the fencing cost.

Question 7

A rectangular box with a square base and no top is to have a volume of 500 cm^3 . What dimensions will minimize the surface area?



Open-top square box

Question 7 - Solution

Let: - x = side length of the square base (in cm) - h = height of the box (in cm)

Volume constraint:

$$V = x^2h = 500 \Rightarrow h = \frac{500}{x^2}$$

Surface area to minimize: - One bottom square: x^2 - Four sides: $4(xh)$

$$A(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{500}{x^2} = x^2 + \frac{2000}{x}$$

Take derivative:

$$A'(x) = 2x - \frac{2000}{x^2}$$

Set derivative to zero:

$$2x = \frac{2000}{x^2} \Rightarrow 2x^3 = 2000 \Rightarrow x^3 = 1000 \Rightarrow x = 10$$

Find h :

$$h = \frac{500}{x^2} = \frac{500}{100} = 5$$

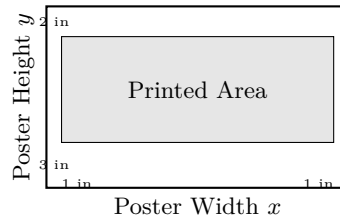
Answer:

$x = 10 \text{ cm}, \quad h = 5 \text{ cm}$

These dimensions minimize the surface area.

Question 8

A poster is to have an area of 192 in^2 . It must have 1-inch margins on the sides, a 2-inch margin at the top, and a 3-inch margin at the bottom. What dimensions of the poster will allow the largest printed area?



Question 8 - Solution

Let x be poster width and y its height. Then $xy = 192$, and positive printed dimensions require $2 < x < 192/5$.

$$A(x) = (x - 2)(192/x - 5) = 202 - 5x - 384/x.$$

$$A'(x) = -5 + 384/x^2 = 0 \implies x = \sqrt{384/5}.$$

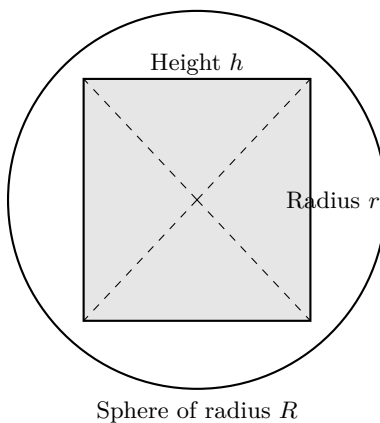
Since $A''(x) = -768/x^3 < 0$, this is the unique maximum.

$x = \sqrt{384/5} \approx 8.76 \text{ in}, \quad y = \sqrt{480} \approx 21.91 \text{ in}.$
--

The exact width, rather than its rounded value, is used to compute the height.

Question 9

A cylinder is inscribed in a sphere of radius R . Find the dimensions (radius and height) of the cylinder that has the maximum volume.



Question 9 - Solution

Let the cylinder have: - Radius: r - Height: h

It is inscribed in a sphere of radius R , so by the Pythagorean Theorem applied to the right triangle formed from the center of the sphere to the top edge of the cylinder:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

Volume of the cylinder:

$$V = \pi r^2 h$$

From the constraint:

$$r^2 = R^2 - \left(\frac{h}{2}\right)^2 = R^2 - \frac{h^2}{4}$$

Substitute into the volume:

$$V(h) = \pi \left(R^2 - \frac{h^2}{4}\right) h = \pi \left(R^2 h - \frac{h^3}{4}\right)$$

Differentiate and maximize:

$$V'(h) = \pi \left(R^2 - \frac{3h^2}{4}\right)$$

Set $V'(h) = 0$:

$$R^2 - \frac{3h^2}{4} = 0 \Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}$$

Then:

$$r^2 = R^2 - \left(\frac{h}{2}\right)^2 = R^2 - \frac{1}{4} \cdot \frac{4R^2}{3} = R^2 - \frac{R^2}{3} = \frac{2R^2}{3} \Rightarrow r = \sqrt{\frac{2}{3}}R$$

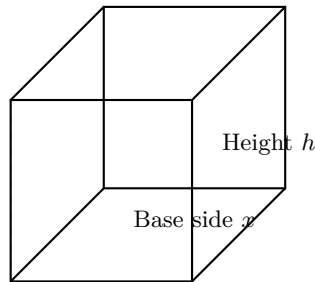
Answer:

$$\boxed{h = \frac{2R}{\sqrt{3}}, \quad r = \sqrt{\frac{2}{3}}R}$$

These are the dimensions of the inscribed cylinder with maximum volume.

Question 10

A box with a square base and an open top must have a volume of V cubic units. Find the dimensions of the box that minimize its surface area.



Question 10 - Solution

Let: - x = length of the base side (square base) - h = height of the box - Volume: $V = x^2h \Rightarrow h = \frac{V}{x^2}$

Surface Area (SA): The box has no top, so:

$$A(x) = \text{Base} + 4 \text{ sides} = x^2 + 4xh$$

$$A(x) = x^2 + 4x \left(\frac{V}{x^2} \right) = x^2 + \frac{4V}{x}$$

Minimize $A(x)$:

Differentiate:

$$A'(x) = 2x - \frac{4V}{x^2}$$

Set $A'(x) = 0$:

$$2x - \frac{4V}{x^2} = 0 \Rightarrow 2x^3 = 4V \Rightarrow x^3 = 2V \Rightarrow x = \sqrt[3]{2V}$$

Then:

$$h = \frac{V}{x^2} = \frac{V}{\left(\sqrt[3]{2V}\right)^2} = \frac{V}{\sqrt[3]{4V^2}} = \sqrt[3]{\frac{V}{4}}$$

Answer:

$x = \sqrt[3]{2V}, \quad h = \sqrt[3]{\frac{V}{4}}$

These dimensions give the minimum surface area for the box.