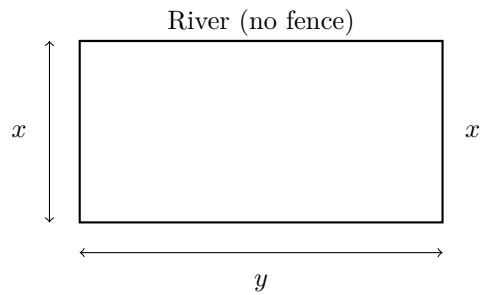


Question 1

A rectangular garden is to be fenced on three sides (two widths and one length), with the fourth side along a straight river needing no fence. The area of the garden must be 120 m^2 . What dimensions will minimize the amount of fencing used?

Clearly define variables, write the function to be minimized, and find the minimum using calculus.

Diagram:



Question 1 - Solution

Let:

- x = width (perpendicular to the river) - y = length (along the river)

Given: - Area constraint: $A = xy = 120 \Rightarrow y = \frac{120}{x}$ - Fence needed: two widths and one length
 $\Rightarrow F = 2x + y$

Substitute into fencing function:

$$F(x) = 2x + \frac{120}{x}$$

Minimize $F(x)$:

Take derivative:

$$F'(x) = 2 - \frac{120}{x^2}$$

Set $F'(x) = 0$:

$$2 = \frac{120}{x^2} \Rightarrow x^2 = 60 \Rightarrow x = \sqrt{60} = 2\sqrt{15}$$

Then:

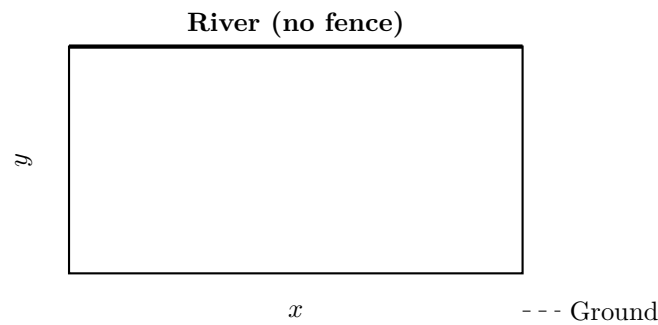
$$y = \frac{120}{x} = \frac{120}{2\sqrt{15}} = \frac{60}{\sqrt{15}} = \frac{60\sqrt{15}}{15} = 4\sqrt{15}$$

Answer: Width = $\boxed{2\sqrt{15} \text{ m}}$, Length = $\boxed{4\sqrt{15} \text{ m}}$

Question 2**Problem:**

A rectangular field is to be enclosed on three sides with fencing (the fourth side lies along a river and requires no fence). The total length of fencing available is 240 meters.

- (a) What dimensions will maximize the area of the field?
- (b) What is the maximum area?

Diagram:

Question 2 - Solution

Let: - x = length parallel to the river (no fence needed) - y = width (two sides to be fenced)

From the constraint:

$$x + 2y = 240 \quad \Rightarrow \quad x = 240 - 2y$$

The area is:

$$A = x \cdot y = (240 - 2y)y = 240y - 2y^2$$

Differentiate:

$$\frac{dA}{dy} = 240 - 4y$$

Set derivative to zero:

$$240 - 4y = 0 \quad \Rightarrow \quad y = 60$$

Back-substitute:

$$x = 240 - 2(60) = 120$$

(a) Optimal Dimensions:

$$x = 120 \text{ m}, \quad y = 60 \text{ m}$$

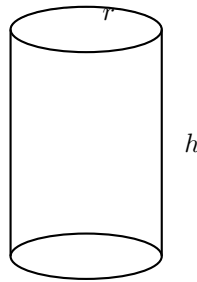
(b) Maximum Area:

$$A = 120 \times 60 = 7200 \text{ m}^2$$

Question 3**Problem:**

A cylindrical can is to be made to hold a volume of 1000 cm^3 . The material for the top and bottom costs twice as much per square centimeter as the material for the side.

- (a) Find the dimensions (radius and height) of the can that minimize the cost of the material.
- (b) What is the minimum total cost in terms of material area?

Diagram:

Question 3 - Solution

Let: - r : radius of the cylinder - h : height of the cylinder

Volume constraint:

$$V = \pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}$$

Cost: - Side area = $2\pi r h$, unit cost = 1 - Top and bottom area = $2\pi r^2$, unit cost = 2

Total cost C (proportional to material area):

$$C = 2\pi r h + 2 \cdot 2\pi r^2 = 2\pi r h + 4\pi r^2$$

Substitute h :

$$C(r) = 2\pi r \cdot \frac{1000}{\pi r^2} + 4\pi r^2 = \frac{2000}{r} + 4\pi r^2$$

Differentiate:

$$C'(r) = -\frac{2000}{r^2} + 8\pi r$$

Set derivative to 0:

$$-\frac{2000}{r^2} + 8\pi r = 0 \Rightarrow 8\pi r^3 = 2000 \Rightarrow r^3 = \frac{250}{\pi}$$

$$r = \sqrt[3]{\frac{250}{\pi}}, \quad h = \frac{1000}{\pi r^2}$$

(a) Optimal Dimensions:

$$\boxed{r = \sqrt[3]{\frac{250}{\pi}}, \quad h = \frac{1000}{\pi r^2}}$$

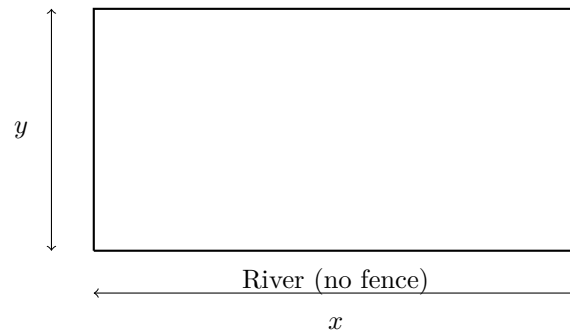
(b) Minimum Cost (in material area units):

$$\boxed{C = \frac{2000}{r} + 4\pi r^2} \quad \text{where } r = \sqrt[3]{\frac{250}{\pi}}$$

Question 4**Problem:**

A farmer wants to build a rectangular fence next to a straight river. No fencing is needed along the river. The farmer has 240 meters of fencing available.

- (a) What dimensions will maximize the area of the enclosure?
- (b) What is the maximum area that can be enclosed?

Diagram:

Question 4 - Solution

Let: - x : length of the fence parallel to the river (only 1 needed) - y : width of the fence perpendicular to the river (2 needed)

Fencing constraint:

$$x + 2y = 240 \quad \Rightarrow \quad x = 240 - 2y$$

Area:

$$A = x \cdot y = (240 - 2y)y = 240y - 2y^2$$

Differentiate:

$$A'(y) = 240 - 4y$$

Set derivative to 0:

$$240 - 4y = 0 \quad \Rightarrow \quad y = 60$$

$$x = 240 - 2(60) = 120$$

(a) Optimal Dimensions:

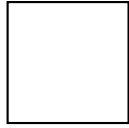
$$x = 120 \text{ m}, \quad y = 60 \text{ m}$$

(b) Maximum Area:

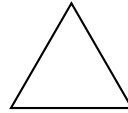
$$A = x \cdot y = 120 \cdot 60 = 7200 \text{ m}^2$$

Question 5

A square and an equilateral triangle are to be constructed from a single wire 100 cm long. The square and the triangle do not overlap. How much wire should be used for each shape in order to minimize the total enclosed area?



Square



Equilateral Triangle

Question 5 - Solution

Let:

$$x = \text{length of wire used for the square, } 0 < x < 100$$

Then:

$$100 - x = \text{length of wire used for the triangle}$$

Step 1: Expressions for Area

Square:

Let $s = \frac{x}{4}$ be the side of the square. Then area of the square is:

$$A_{\text{square}} = s^2 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

Triangle:

Let each side of the triangle be $t = \frac{100-x}{3}$. Then area of equilateral triangle is:

$$A_{\text{triangle}} = \frac{\sqrt{3}}{4}t^2 = \frac{\sqrt{3}}{4} \left(\frac{100-x}{3}\right)^2 = \frac{\sqrt{3}}{36}(100-x)^2$$

Total Area:

$$A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(100-x)^2$$

Step 2: Minimize Total Area

Differentiate:

$$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(100-x)$$

Set $A'(x) = 0$:

$$\frac{1}{8}x = \frac{\sqrt{3}}{18}(100-x) \Rightarrow 18x = 8\sqrt{3}(100-x) \Rightarrow 18x = 800\sqrt{3} - 8\sqrt{3}x \Rightarrow (18+8\sqrt{3})x = 800\sqrt{3} \Rightarrow x = \frac{800\sqrt{3}}{18+8\sqrt{3}}$$

Rationalizing denominator:

$$x = \frac{800\sqrt{3}(18-8\sqrt{3})}{(18+8\sqrt{3})(18-8\sqrt{3})} = \frac{800\sqrt{3}(18-8\sqrt{3})}{324-192} = \frac{800\sqrt{3}(18-8\sqrt{3})}{132}$$

Approximate:

$$x \approx 43.5 \Rightarrow \text{Use about } \boxed{43.5 \text{ cm for the square}}, \boxed{56.5 \text{ cm for the triangle}}$$

Step 3: Justification for Minimum

We compute the second derivative:

$$A''(x) = \frac{1}{8} + \frac{\sqrt{3}}{18} > 0$$

Since $A''(x) > 0$, this critical point gives a local minimum.

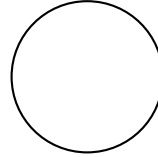
The wire distribution minimizes total area.

Question 6

A 100 cm long wire is to be cut into two pieces. One piece is bent to form a square, and the other is bent to form a circle. How should the wire be cut to minimize the total enclosed area?



Square



Circle

Question 6 - Solution

Let:

x = length used for the square, $0 < x < 100 \Rightarrow 100 - x$ = length used for the circle

Step 1: Area Expressions

Square:

$$\text{Perimeter} = x \Rightarrow \text{Side} = \frac{x}{4}, \quad A_{\text{square}} = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

Circle:

$$\text{Circumference} = 100 - x = 2\pi r \Rightarrow r = \frac{100 - x}{2\pi} \Rightarrow A_{\text{circle}} = \pi r^2 = \pi \left(\frac{100 - x}{2\pi}\right)^2 = \frac{(100 - x)^2}{4\pi}$$

Total Area:

$$A(x) = \frac{x^2}{16} + \frac{(100 - x)^2}{4\pi}$$

Step 2: Minimize Area

Differentiate:

$$A'(x) = \frac{1}{8}x - \frac{1}{2\pi}(100 - x)$$

Set $A'(x) = 0$:

$$\frac{1}{8}x = \frac{1}{2\pi}(100 - x) \Rightarrow \pi x = 4(100 - x) \Rightarrow \pi x = 400 - 4x \Rightarrow x(\pi + 4) = 400 \Rightarrow x = \frac{400}{\pi + 4}$$

Step 3: Approximate and Justify Minimum

$$x \approx \frac{400}{3.1416 + 4} = \frac{400}{7.1416} \approx 56.0$$

$$\boxed{x \approx 56.0 \text{ cm for the square}}, \quad \boxed{44.0 \text{ cm for the circle}}$$

Second derivative:

$$A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0 \Rightarrow \text{This critical point is a minimum.}$$

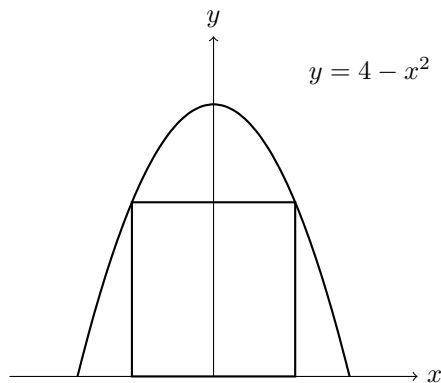
$$\boxed{\text{This wire division minimizes the total area.}}$$

Question 7

Find the dimensions of the rectangle of maximum area that can be inscribed under the parabola

$$y = 4 - x^2$$

with its base lying along the x-axis and the top two corners touching the curve.



Question 7 - Solution

Let the rectangle extend from $-x$ to x , so its width is $2x$, and its height is the y-value of the curve:

$$\text{Height} = y = 4 - x^2$$

Step 1: Area Function

$$A(x) = \text{width} \times \text{height} = 2x(4 - x^2) = 8x - 2x^3$$

Step 2: Maximize the Area

Differentiate:

$$A'(x) = 8 - 6x^2$$

Set $A'(x) = 0$:

$$8 - 6x^2 = 0 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}} \approx 1.1547$$

Step 3: Dimensions of the Rectangle

$$\text{Width} = 2x = \frac{4}{\sqrt{3}} \approx 2.309$$

$$\text{Height} = 4 - x^2 = 4 - \frac{4}{3} = \frac{8}{3} \approx 2.667$$

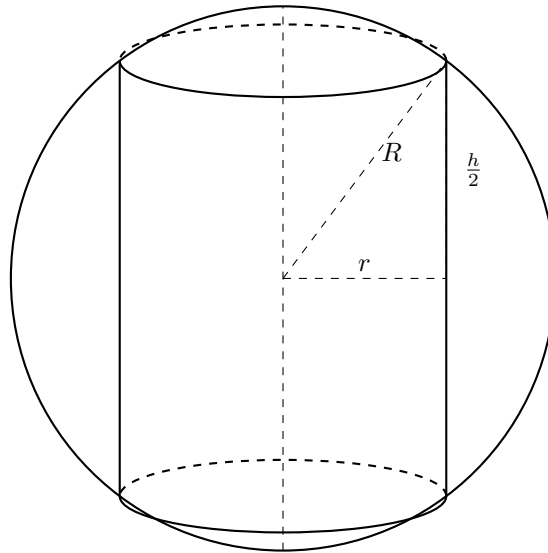
Maximum Area Rectangle Dimensions: Width ≈ 2.309 , Height ≈ 2.667

Step 4: Confirm Maximum

$$A''(x) = -12x < 0 \quad \text{at } x > 0 \Rightarrow \text{maximum confirmed.}$$

Question 8

A cylinder is inscribed in a sphere of radius R . Find the maximum possible volume of such a cylinder and determine its dimensions.



Cylinder inscribed in a sphere of radius R

Question 8 - Solution

Let the height of the cylinder be h and its radius be r . From the diagram, the diagonal of the right triangle gives the constraint:

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

Step 1: Volume Function

$$V = \pi r^2 h = \pi r^2 \cdot 2\sqrt{R^2 - r^2} = 2\pi r^2 \sqrt{R^2 - r^2}$$

Step 2: Maximize $V(r)$

Let:

$$V(r) = 2\pi r^2 (R^2 - r^2)^{1/2}$$

Differentiate using the product and chain rule:

$$V'(r) = 2\pi \left[2r(R^2 - r^2)^{1/2} - \frac{r^3}{(R^2 - r^2)^{1/2}} \right] = \frac{2\pi}{(R^2 - r^2)^{1/2}} (2r(R^2 - r^2) - r^3)$$

Set $V'(r) = 0$:

$$2r(R^2 - r^2) - r^3 = 0 \Rightarrow r(2R^2 - 3r^2) = 0 \Rightarrow r = 0 \text{ or } r^2 = \frac{2R^2}{3}$$

Step 3: Dimensions

$$r = \sqrt{\frac{2}{3}}R, \quad h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$$

Step 4: Max Volume

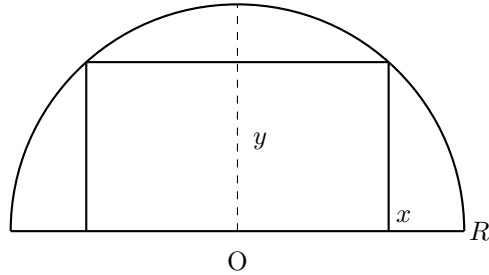
$$V = \pi r^2 h = \pi \cdot \left(\frac{2R^2}{3}\right) \cdot \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

Radius: $r = \sqrt{\frac{2}{3}}R$
Height: $h = \frac{2R}{\sqrt{3}}$
Max Volume: $V = \frac{4\pi R^3}{3\sqrt{3}}$

Question 1

A rectangle is inscribed in a semicircle of radius R , with the rectangle's base lying along the diameter of the semicircle.

Find the dimensions of the rectangle that has the maximum area.



Rectangle inscribed in a semicircle

Question 1 - Solution

Let the coordinates of the top-right corner of the rectangle be (x, y) . Since the rectangle is symmetric and lies in a semicircle of radius R centered at the origin:

$$x^2 + y^2 = R^2 \quad (\text{semicircle equation}) \Rightarrow y = \sqrt{R^2 - x^2}$$

Step 1: Area of Rectangle

Since the base is $2x$ and the height is y , area is:

$$A(x) = 2x \cdot y = 2x\sqrt{R^2 - x^2}$$

Step 2: Maximize Area

$$A(x) = 2x\sqrt{R^2 - x^2}$$

Differentiate using the product rule:

$$A'(x) = 2\sqrt{R^2 - x^2} + 2x \cdot \frac{-x}{\sqrt{R^2 - x^2}} = \frac{2(R^2 - x^2) - 2x^2}{\sqrt{R^2 - x^2}} = \frac{2R^2 - 4x^2}{\sqrt{R^2 - x^2}}$$

Set $A'(x) = 0$:

$$2R^2 - 4x^2 = 0 \Rightarrow x^2 = \frac{R^2}{2} \Rightarrow x = \frac{R}{\sqrt{2}}, \quad y = \sqrt{R^2 - x^2} = \frac{R}{\sqrt{2}}$$

Step 3: Max Area and Dimensions

$$\text{Width} = 2x = \sqrt{2}R, \quad \text{Height} = y = \frac{R}{\sqrt{2}}, \quad \text{Area} = 2x \cdot y = R^2$$

Maximum area: $A = R^2$

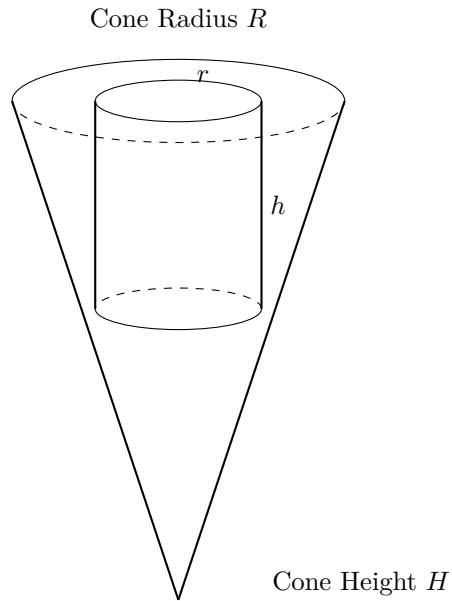
Width: $2x = \sqrt{2}R$

Height: $y = \frac{R}{\sqrt{2}}$

Question 10

A right circular cylinder is inscribed in a right circular cone of height H and base radius R , such that the cylinder shares the same central axis as the cone.

Find the dimensions of the cylinder that maximize its volume.



Question 10 - Solution

Let the height of the cylinder be h , and its base radius be r . We want to maximize:

$$V = \pi r^2 h$$

Step 1: Relationship between r and h

From similar triangles (cone vs cylinder section), we have:

$$\frac{r}{H-h} = \frac{R}{H} \Rightarrow r = \frac{R(H-h)}{H}$$

Substitute into volume:

$$V(h) = \pi \left(\frac{R(H-h)}{H} \right)^2 h = \pi \cdot \frac{R^2(H-h)^2 h}{H^2}$$

Step 2: Maximize $V(h)$

Let's simplify:

$$V(h) = \frac{\pi R^2}{H^2} (H-h)^2 h$$

Let $k = \frac{\pi R^2}{H^2}$, then:

$$V(h) = k(H-h)^2 h$$

Use the product rule:

$$V'(h) = k [2(H-h)(-1) \cdot h + (H-h)^2] = k [-2h(H-h) + (H-h)^2]$$

$$= k(H-h)[(H-h) - 2h] = k(H-h)(H-3h)$$

Set $V'(h) = 0$:

$$(H-h)(H-3h) = 0 \Rightarrow h = H \text{ or } h = \frac{H}{3}$$

But $h = H$ gives 0 volume. So maximum occurs at $\boxed{h = \frac{H}{3}}$

Substitute into r :

$$r = \frac{R(H-H/3)}{H} = \frac{2R}{3}$$

Step 3: Final Answer

Height of cylinder: $h = \frac{H}{3}$
Radius of cylinder: $r = \frac{2R}{3}$
Maximum Volume: $V = \pi r^2 h = \pi \cdot \left(\frac{2R}{3} \right)^2 \cdot \frac{H}{3} = \frac{4\pi R^2 H}{27}$