

Question 1

Let

$$f(x) = x^2 + 2x$$

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$.
- (b) Find all numbers $c \in (1, 4)$ that satisfy the conclusion of the Mean Value Theorem.

Question 1 - Solution

We are given

$$f(x) = x^2 + 2x$$

(a) Verifying the hypotheses

The function $f(x)$ is a polynomial, so it is continuous on the closed interval $[1, 4]$ and differentiable on the open interval $(1, 4)$.

Therefore, the hypotheses of the Mean Value Theorem are satisfied.

(b) Applying the Mean Value Theorem

First compute the derivative:

$$f'(x) = 2x + 2$$

Next compute the average rate of change of f on $[1, 4]$:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{(16 + 8) - (1 + 2)}{3} = \frac{21}{3} = 7$$

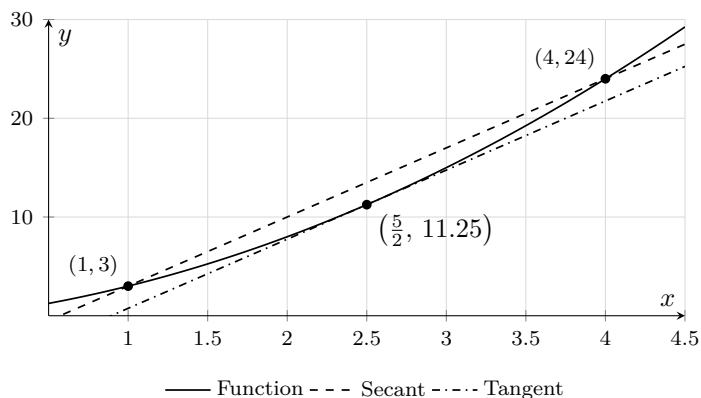
Set the derivative equal to this value:

$$2c + 2 = 7 \Rightarrow 2c = 5 \Rightarrow c = \frac{5}{2}$$

Thus, the number that satisfies the conclusion of the Mean Value Theorem is

$$c = \frac{5}{2}$$

Graph of $f(x)$, the secant line, and the tangent line at c



Question 2**Problem:**

Let

$$f(x) = \ln(x), \quad \text{for } x \in [1, 4]$$

- (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval.
- (b) Find all values of $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

Question 2 - Solution

We are given:

$$f(x) = \ln(x)$$

(a) **Check MVT hypotheses:**

1. $f(x)$ is continuous on $[1, 4]$: Yes, $\ln(x)$ is continuous for $x > 0$, so continuous on $[1, 4]$
 2. $f(x)$ is differentiable on $(1, 4)$: Yes, $\ln(x)$ is differentiable for $x > 0$, so differentiable on $(1, 4)$
- The Mean Value Theorem applies.

(b) **Find the value(s) of c :**

Compute average rate of change:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\ln(4) - \ln(1)}{3} = \frac{\ln(4)}{3} = \frac{2\ln(2)}{3}$$

Now compute $f'(x)$:

$$f'(x) = \frac{1}{x}$$

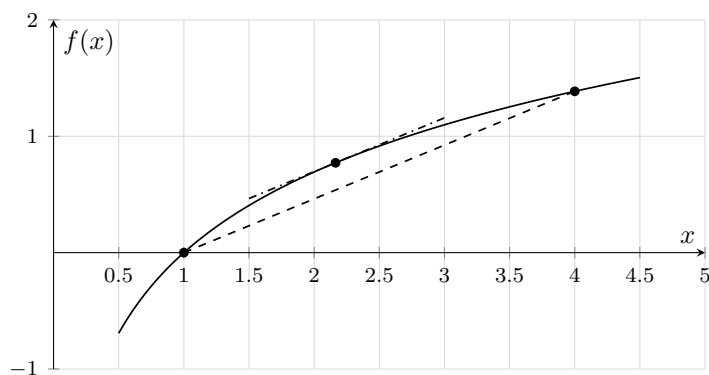
Set:

$$f'(c) = \frac{1}{c} = \frac{2\ln(2)}{3} \Rightarrow c = \frac{3}{2\ln(2)}$$

Answer:

$$c = \frac{3}{2\ln(2)} \approx 2.164$$

Graph of $f(x) = \ln(x)$ on $[1, 4]$:



Question 2**Problem:**

Let

$$f(x) = x^3 - 3x^2 + 2x, \quad \text{on the interval } [1, 3]$$

- (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on $[1, 3]$.
- (b) Find all values of $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

Question 2 - Solution

We are given:

$$f(x) = x^3 - 3x^2 + 2x$$

(a) **Hypotheses of the MVT:**

- $f(x)$ is a polynomial continuous on $[1, 3]$ - $f(x)$ is differentiable on $(1, 3)$
Therefore, MVT applies.

(b) **Apply the MVT:**

Compute:

$$f(3) = 27 - 27 + 6 = 6, \quad f(1) = 1 - 3 + 2 = 0$$

$$\text{Average rate of change} = \frac{6 - 0}{3 - 1} = \frac{6}{2} = 3$$

Compute derivative:

$$f'(x) = 3x^2 - 6x + 2$$

Set $f'(c) = 3$:

$$3c^2 - 6c + 2 = 3 \Rightarrow 3c^2 - 6c - 1 = 0$$

Solve:

$$c = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36 + 12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm 4\sqrt{3}}{6} = \frac{3 \pm 2\sqrt{3}}{3}$$

Answer:

$$\boxed{c = \frac{3 \pm 2\sqrt{3}}{3}} \quad \text{Only the value(s) in } (1, 3) \text{ are valid.}$$

Since:

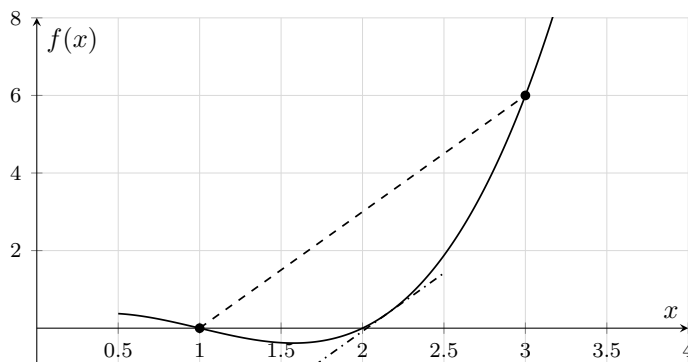
$$\frac{3 - 2\sqrt{3}}{3} \approx 0.845 < 1 \quad (\text{not valid})$$

$$\frac{3 + 2\sqrt{3}}{3} \approx 2.155 \in (1, 3)$$

Final answer:

$$\boxed{c = \frac{3 + 2\sqrt{3}}{3}}$$

Graph of $f(x) = x^3 - 3x^2 + 2x$ on $[1, 3]$:



Question 4**Problem:**

Let

$$f(x) = \ln(x^2 + 1) \quad \text{on the interval } [-1, 2].$$

- (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on $[-1, 2]$.
- (b) Find all values of $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$

Question 4 - Solution

The function $f(x) = \ln(1 + x^2)$ is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$, so the Mean Value Theorem applies.

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{\ln(5/2)}{3}, \quad f'(c) = \frac{2c}{1 + c^2}.$$

Let $L = \ln(5/2)$. Then $Lc^2 - 6c + L = 0$, so

$$c = \frac{3 \pm \sqrt{9 - L^2}}{L}.$$

The plus root exceeds 2. The only admissible value is

$$c = \frac{3 - \sqrt{9 - [\ln(5/2)]^2}}{\ln(5/2)} \approx 0.156453.$$

Question 5**Problem:**

Let

$$f(x) = \frac{x}{x+2} \quad \text{on the interval } [1, 4].$$

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on $[1, 4]$.
- (b) Find all values $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Question 5 - Solution

We are given:

$$f(x) = \frac{x}{x+2}$$

(a) **Check MVT conditions:**

- $f(x)$ is a rational function with denominator $x+2 \neq 0$ on $[1, 4]$ continuous on $[1, 4]$. - It is differentiable on $(1, 4)$ as it's smooth where defined.

MVT applies.

(b) **Compute average rate of change:**

$$f(1) = \frac{1}{3}, \quad f(4) = \frac{4}{6} = \frac{2}{3}$$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\frac{2}{3} - \frac{1}{3}}{3} = \frac{1}{9}$$

Find derivative:

Use quotient rule:

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

Set:

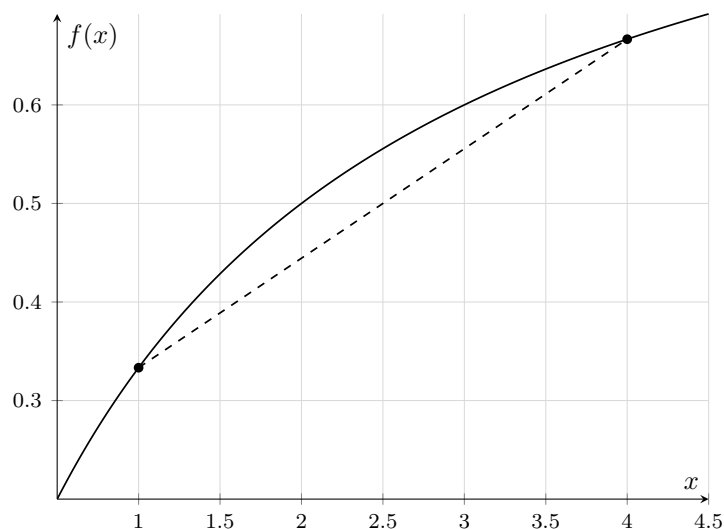
$$\frac{2}{(c+2)^2} = \frac{1}{9} \Rightarrow (c+2)^2 = 18 \Rightarrow c+2 = \pm\sqrt{18} = \pm 3\sqrt{2} \Rightarrow c = -2 \pm 3\sqrt{2}$$

Now, only one of these is in the interval $(1, 4)$:

$$c = -2 + 3\sqrt{2} \approx -2 + 4.24 = 2.24$$

$$\boxed{c = -2 + 3\sqrt{2}} \text{ is in } (1, 4)$$

Graph of $f(x)$ and secant line:



Question 6**Problem:**

Let

$$f(x) = \ln(x^2 + 1)$$

on the interval $[-1, 2]$.

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem on the interval.
- (b) Find all values of $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$

Question 6 - Solution

The function $f(x) = \ln(1 + x^2)$ is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$, so the Mean Value Theorem applies.

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{\ln(5/2)}{3}, \quad f'(c) = \frac{2c}{1 + c^2}.$$

Let $L = \ln(5/2)$. Then $Lc^2 - 6c + L = 0$, so

$$c = \frac{3 \pm \sqrt{9 - L^2}}{L}.$$

The plus root exceeds 2. The only admissible value is

$$c = \frac{3 - \sqrt{9 - [\ln(5/2)]^2}}{\ln(5/2)} \approx 0.156453.$$

Question 7**Problem:**

Let

$$f(x) = \frac{x}{x+1}$$

on the interval $[1, 4]$.

- (a) Show that the hypotheses of the Mean Value Theorem are satisfied.
- (b) Find all values of $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Question 7 - Solution

We are given:

$$f(x) = \frac{x}{x+1}$$

(a) **MVT Conditions:**

- $f(x)$ is continuous on $[1, 4]$: The function is a rational function with no discontinuities on this interval.
- $f(x)$ is differentiable on $(1, 4)$: Differentiable as a quotient of differentiable functions, and denominator never zero.

MVT applies.

(b) **Compute the average rate of change:**

$$\begin{aligned} f(1) &= \frac{1}{2}, & f(4) &= \frac{4}{5} \\ \frac{f(4) - f(1)}{4 - 1} &= \frac{\frac{4}{5} - \frac{1}{2}}{3} = \frac{\frac{8-5}{10}}{3} = \frac{3}{10} \cdot \frac{1}{3} = \frac{1}{10} \end{aligned}$$

Find derivative:

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

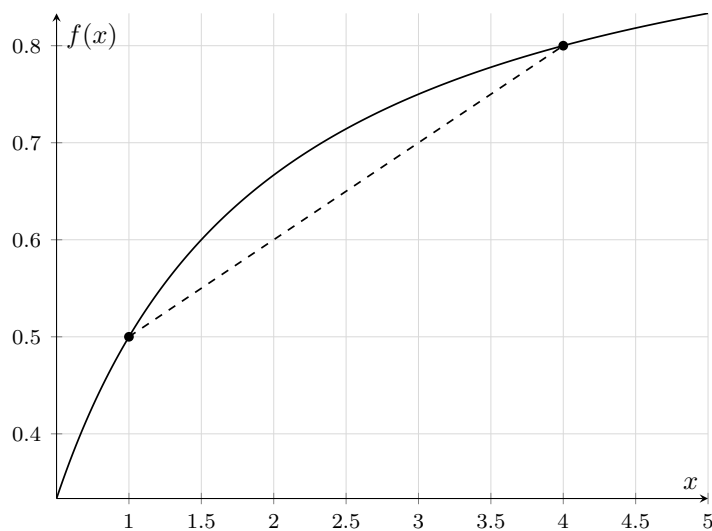
Set:

$$\frac{1}{(c+1)^2} = \frac{1}{10} \Rightarrow (c+1)^2 = 10 \Rightarrow c+1 = \sqrt{10} \Rightarrow c = \sqrt{10} - 1$$

Since $\sqrt{10} \approx 3.16$, we find $c \approx 2.16 \in (1, 4)$

$c = \sqrt{10} - 1$

Graph of $f(x)$ and Secant Line:



Question 8**Problem:**

Let

$$f(x) = \sin(x)$$

on the interval $[0, \pi]$.

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem.
- (b) Find all numbers $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}.$$

Question 8 - Solution

We are given:

$$f(x) = \sin(x)$$

(a) MVT Conditions:

- $f(x)$ is continuous on $[0, \pi]$: Sine is continuous everywhere.
 - $f(x)$ is differentiable on $(0, \pi)$: Sine is differentiable everywhere.
- MVT applies.

(b) Apply MVT:

Average rate of change:

$$f(\pi) = \sin(\pi) = 0, \quad f(0) = \sin(0) = 0$$

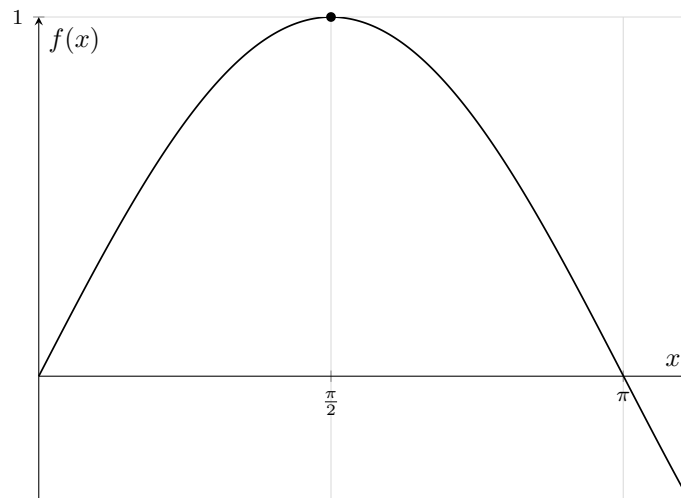
$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

We want:

$$f'(c) = \cos(c) = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\boxed{c = \frac{\pi}{2}}$$

Graph of $f(x) = \sin(x)$ on $[0, \pi]$:



Question 9**Problem:**

Let

$$f(x) = \frac{1}{x}$$

on the interval $[1, 3]$.

- (a) Verify that $f(x)$ satisfies the hypotheses of the Mean Value Theorem.
- (b) Find all numbers $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

Question 9 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

(a) **Conditions for MVT:**

- $f(x)$ is continuous on $[1, 3]$
 - $f(x)$ is differentiable on $(1, 3)$
- Therefore, MVT applies.

(b) **Apply MVT:**

$$f(3) = \frac{1}{3}, \quad f(1) = 1$$

$$\frac{f(3) - f(1)}{3 - 1} = \frac{\frac{1}{3} - 1}{2} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3}$$

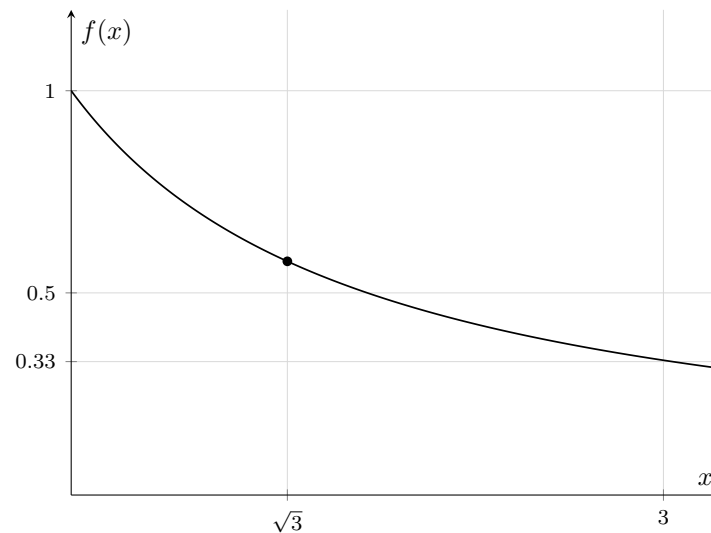
Now compute derivative:

$$f'(x) = \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$$

$$f'(c) = -\frac{1}{c^2} = -\frac{1}{3} \Rightarrow c^2 = 3 \Rightarrow c = \sqrt{3}$$

$$c = \sqrt{3} \in (1, 3)$$

Graph of $f(x) = \frac{1}{x}$ on $[1, 3]$:



Question 10**Problem:**

Let

$$f(x) = \sin(x)$$

on the interval $[0, \pi]$.

- (a) Verify that $f(x)$ satisfies the conditions of the Mean Value Theorem on $[0, \pi]$.
- (b) Find all values of $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}.$$

Question 10 - Solution

We are given:

$$f(x) = \sin(x)$$

(a) **Conditions for MVT:**

- $f(x)$ is continuous on $[0, \pi]$ - $f(x)$ is differentiable on $(0, \pi)$
MVT applies.

(b) **Apply MVT:**

$$\begin{aligned} f(\pi) &= 0, & f(0) &= 0 \\ \frac{f(\pi) - f(0)}{\pi - 0} &= \frac{0 - 0}{\pi} = 0 \end{aligned}$$

We need:

$$f'(c) = \cos(c) = 0 \Rightarrow c = \frac{\pi}{2}$$

$$\boxed{c = \frac{\pi}{2}}$$

Graph of $f(x) = \sin(x)$ on $[0, \pi]$:

