

Question 1

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify all local maxima and minima.
- (c) Determine intervals of concavity and any inflection points.

Question 1 - Solution

We are given

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) First derivative

$$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$$

The critical points are $x = -1$ and $x = 3$.

For $x < -1$, the derivative is positive. For $-1 < x < 3$, the derivative is negative. For $x > 3$, the derivative is positive.

Therefore, the function is increasing on

$$(-\infty, -1) \cup (3, \infty)$$

and decreasing on

$$(-1, 3).$$

(b) Local extrema

At $x = -1$, the function changes from increasing to decreasing, so there is a local maximum.

$$f(-1) = 10$$

At $x = 3$, the function changes from decreasing to increasing, so there is a local minimum.

$$f(3) = -22$$

Local maximum at $(-1, 10)$

Local minimum at $(3, -22)$

(c) Second derivative

$$f''(x) = 6x - 6$$

Setting $f''(x) = 0$ gives $x = 1$.

For $x < 1$, the second derivative is negative, so the graph is concave down. For $x > 1$, the second derivative is positive, so the graph is concave up.

The inflection point occurs at

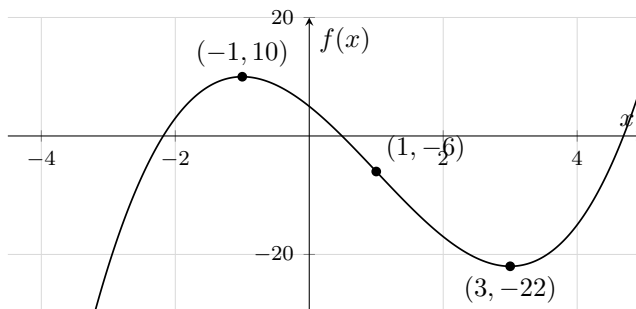
$$f(1) = -6$$

Concave down on $(-\infty, 1)$

Concave up on $(1, \infty)$

Inflection point at $(1, -6)$

Graph of $f(x)$



Question 2

Let

$$f(x) = x^4 - 4x^3$$

- (a) Find the intervals where the graph of f is concave up and concave down.
- (b) Identify any inflection points.

Question 2 - Solution

We are given

$$f(x) = x^4 - 4x^3$$

First derivative

$$f'(x) = 4x^3 - 12x^2$$

Second derivative

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

(a) Concavity

Setting $f''(x) = 0$ gives the critical values

$$x = 0 \quad \text{and} \quad x = 2.$$

For $x < 0$, the second derivative is positive, so the graph is concave up. For $0 < x < 2$, the second derivative is negative, so the graph is concave down. For $x > 2$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\infty, 0) \cup (2, \infty)$$

Concave down on

$$(0, 2)$$

(b) Inflection points

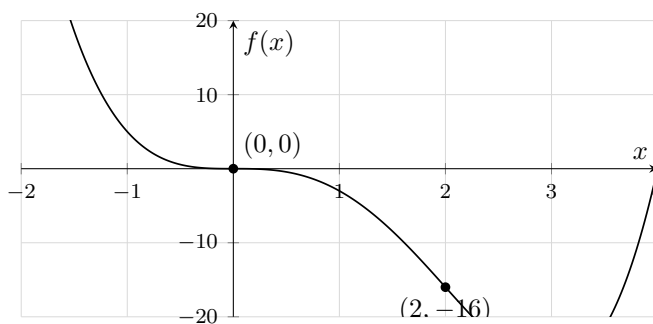
Because the concavity changes at both $x = 0$ and $x = 2$, inflection points occur at these values.

$$f(0) = 0 \quad f(2) = 16 - 32 = -16$$

Inflection points at

$$(0, 0) \quad \text{and} \quad (2, -16)$$

Graph of $f(x)$



Question 3

Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 3 - Solution

We are given

$$f(x) = \frac{x^2}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(2x)(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$ gives

$$f''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ yields

$$1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}.$$

For $|x| < \frac{1}{\sqrt{3}}$, the second derivative is positive, so the graph is concave up.

For $|x| > \frac{1}{\sqrt{3}}$, the second derivative is negative, so the graph is concave down.

Concave up on

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Concave down on

$$\left(-\infty, -\frac{1}{\sqrt{3}} \right) \cup \left(\frac{1}{\sqrt{3}}, \infty \right)$$

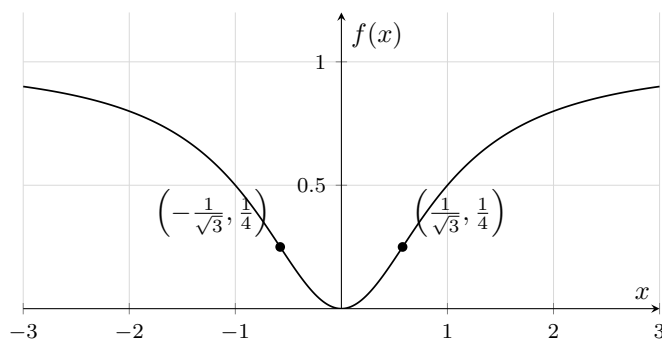
(b) Inflection points

Because the concavity changes at $x = \pm \frac{1}{\sqrt{3}}$, inflection points occur there.

$$f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{1}{4}$$

Inflection points at

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{4} \right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4} \right)$$

Graph of $f(x)$ 

Question 4

Let

$$f(x) = x^3 - 3x$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 4 - Solution

We are given

$$f(x) = x^3 - 3x$$

First derivative

$$f'(x) = 3x^2 - 3$$

Second derivative

$$f''(x) = 6x$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 0.$$

For $x < 0$, the second derivative is negative, so the graph is concave down.

For $x > 0$, the second derivative is positive, so the graph is concave up.

Concave down on

$$(-\infty, 0)$$

Concave up on

$$(0, \infty)$$

(b) Inflection point

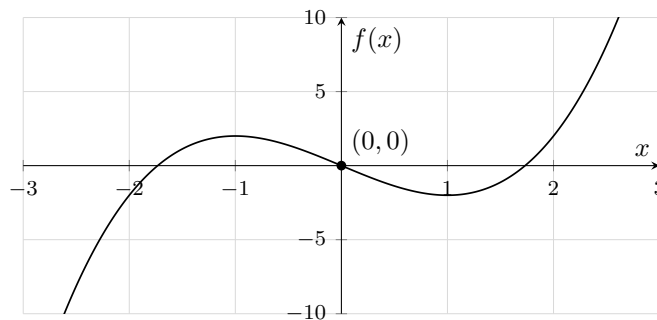
Because the concavity changes at $x = 0$, an inflection point occurs there.

$$f(0) = 0$$

Inflection point at

$$(0, 0)$$

Graph of $f(x)$



Question 5

Let

$$f(x) = \frac{x}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 5 - Solution

We are given

$$f(x) = \frac{x}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{-2x(3 - x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

For $x < -\sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $-\sqrt{3} < x < 0$, the second derivative is positive, so the graph is concave up.

For $0 < x < \sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $x > \sqrt{3}$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$$

Concave down on

$$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

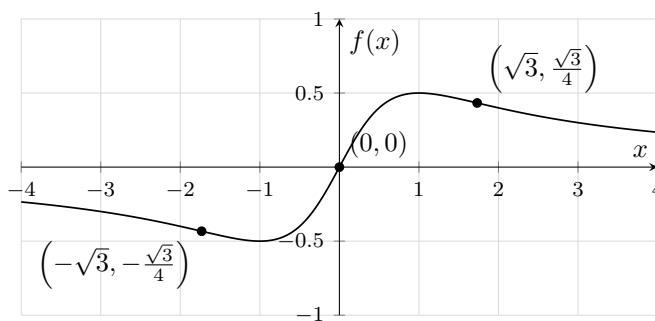
(b) Inflection points

Because the concavity changes at each critical value, inflection points occur at all three values.

$$f(0) = 0, \quad f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4}$$

Inflection points at

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0, 0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$$

Graph of $f(x)$ 

Question 6

Let

$$f(x) = x^4 - 4x^3 + 10$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 6 - Solution

We are given

$$f(x) = x^4 - 4x^3 + 10$$

First derivative

$$f'(x) = 4x^3 - 12x^2$$

Second derivative

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 0 \quad \text{and} \quad x = 2.$$

For $x < 0$, the second derivative is positive, so the graph is concave up.

For $0 < x < 2$, the second derivative is negative, so the graph is concave down.

For $x > 2$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\infty, 0) \cup (2, \infty)$$

Concave down on

$$(0, 2)$$

(b) Inflection points

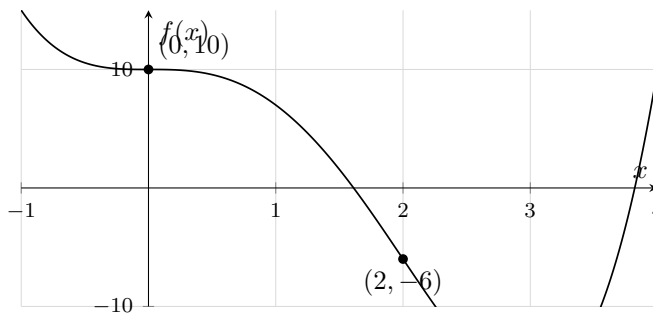
Because the concavity changes at both values, inflection points occur at $x = 0$ and $x = 2$.

$$f(0) = 10, \quad f(2) = -6$$

Inflection points at

$$(0, 10) \quad \text{and} \quad (2, -6)$$

Graph of $f(x)$



Question 7

Let

$$f(x) = x^3 - 6x^2 + 12x - 5$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Find all inflection points.

Question 7 - Solution

We are given

$$f(x) = x^3 - 6x^2 + 12x - 5$$

First derivative

$$f'(x) = 3x^2 - 12x + 12$$

Second derivative

$$f''(x) = 6x - 12$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 2.$$

For $x < 2$, the second derivative is negative, so the graph is concave down.

For $x > 2$, the second derivative is positive, so the graph is concave up.

Concave down on

$$(-\infty, 2)$$

Concave up on

$$(2, \infty)$$

(b) Inflection point

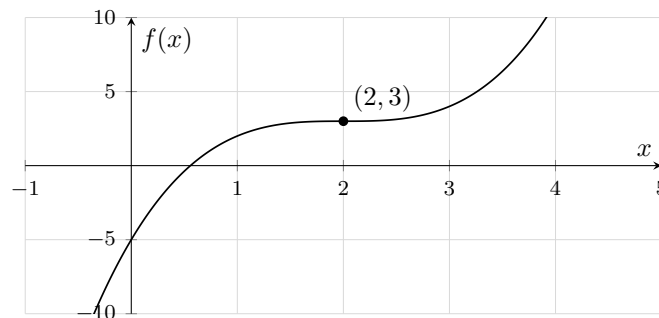
Because the concavity changes at $x = 2$, an inflection point occurs there.

$$f(2) = 3$$

Inflection point at

$$(2, 3)$$

Graph of $f(x)$



Question 8

Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Find all inflection points, if any.

Question 8 - Solution

We are given

$$f(x) = \frac{x^2}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(2x)(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$1 - 3x^2 = 0 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{3}}.$$

For $|x| < \frac{1}{\sqrt{3}}$, the second derivative is positive, so the graph is concave up.

For $|x| > \frac{1}{\sqrt{3}}$, the second derivative is negative, so the graph is concave down.

Concave up on

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Concave down on

$$\left(-\infty, -\frac{1}{\sqrt{3}} \right) \cup \left(\frac{1}{\sqrt{3}}, \infty \right)$$

(b) Inflection points

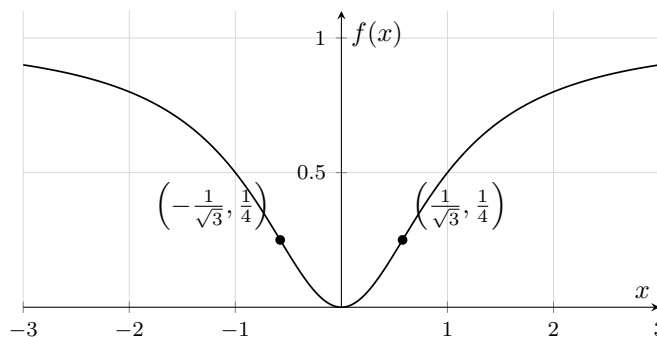
Because the concavity changes at both values, inflection points occur at

$$x = \pm \frac{1}{\sqrt{3}}.$$

$$f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{1}{4}$$

Inflection points at

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{4} \right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4} \right)$$

Graph of $f(x)$ 

Question 9

Let

$$f(x) = x^3 - 3x$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Find all inflection points.

Question 9 - Solution

We are given

$$f(x) = x^3 - 3x$$

First derivative

$$f'(x) = 3x^2 - 3$$

Second derivative

$$f''(x) = 6x$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = 0.$$

For $x < 0$, the second derivative is negative, so the graph is concave down.

For $x > 0$, the second derivative is positive, so the graph is concave up.

Concave down on

$$(-\infty, 0)$$

Concave up on

$$(0, \infty)$$

(b) Inflection point

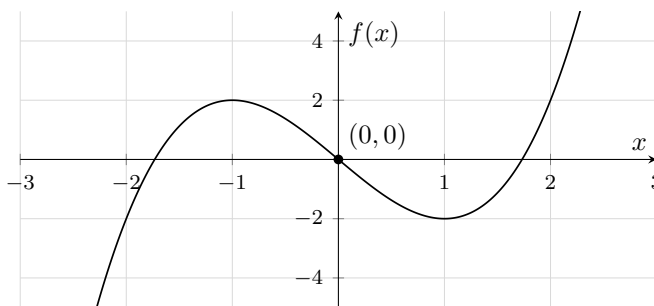
Because the concavity changes at $x = 0$, an inflection point occurs there.

$$f(0) = 0$$

Inflection point at

$$(0, 0)$$

Graph of $f(x)$



Question 10

Let

$$f(x) = \frac{x}{x^2 + 1}$$

- (a) Determine the intervals where f is concave up and concave down.
- (b) Find the inflection points of f , if any.

Question 10 - Solution

We are given

$$f(x) = \frac{x}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{-2x(3 - x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

For $x < -\sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $-\sqrt{3} < x < 0$, the second derivative is positive, so the graph is concave up.

For $0 < x < \sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $x > \sqrt{3}$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$$

Concave down on

$$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

(b) Inflection points

Because the concavity changes at all three values, inflection points occur at each one.

$$f(0) = 0, \quad f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4}$$

Inflection points at

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0, 0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$$

Graph of $f(x)$ 