

Question 1

Let:

$$f(x) = x^4 - 4x^3 + 6x^2$$

- (a) Find the intervals where $f(x)$ is concave up or concave down.
- (b) Determine the inflection points of $f(x)$, if any.

Question 1 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6x^2$$

Step 1: Find the second derivative

$$f'(x) = 4x^3 - 12x^2 + 12x$$

$$f''(x) = 12x^2 - 24x + 12 = 12(x - 1)^2$$

(a) Since $f''(x) = 12(x - 1)^2 \geq 0$, the second derivative is never negative.

For all $x \neq 1$, $f''(x) > 0$, so the function is concave up.

At $x = 1$, the second derivative is zero, but the sign of $f''(x)$ does not change.

Answer

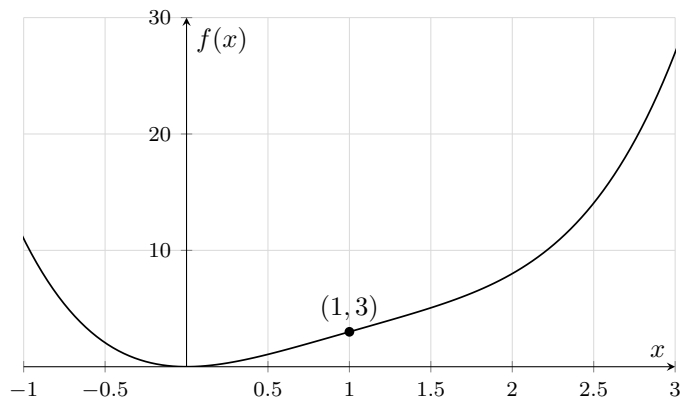
Concave up on

Concave down:

(b) Because the concavity does not change at $x = 1$, there is no inflection point.

Answer

Graph of the function



Question 2**Problem**

Given the function

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the intervals where the function is increasing and decreasing.
- (b) Identify and classify all local extrema.

Question 2 - Solution

We are given

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) **First derivative**

$$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$$

The critical points are

$$x = -1 \quad \text{and} \quad x = 3$$

We test the sign of $f'(x)$.

For $x < -1$, $f'(x) > 0$. For $-1 < x < 3$, $f'(x) < 0$. For $x > 3$, $f'(x) > 0$.

Therefore, the function is increasing on $(-\infty, -1) \cup (3, \infty)$

and decreasing on $(-1, 3)$.

(b) **Local extrema**

Since the function changes from increasing to decreasing at $x = -1$, there is a local maximum:

$$f(-1) = 10$$

Since the function changes from decreasing to increasing at $x = 3$, there is a local minimum:

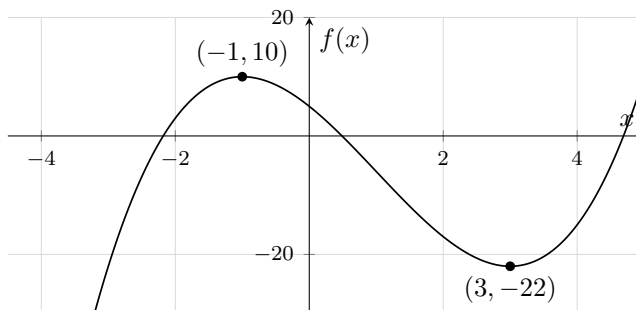
$$f(3) = -22$$

Answer

Local maximum at $(-1, 10)$

Local minimum at $(3, -22)$

Graph of $f(x)$



Question 3**Problem**

Consider the function

$$f(x) = \frac{x^2 - 4x}{x + 1}$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum or minimum points.

Question 3 - Solution

We are given

$$f(x) = \frac{x^2 - 4x}{x + 1} = \frac{x(x - 4)}{x + 1}, \quad x \neq -1$$

(a) First derivative

Using the quotient rule,

$$f'(x) = \frac{(x + 1)(2x - 4) - (x^2 - 4x)}{(x + 1)^2} = \frac{x^2 + 2x - 4}{(x + 1)^2}.$$

Setting the numerator equal to zero gives

$$x^2 + 2x - 4 = 0 \Rightarrow x = -1 \pm \sqrt{5}.$$

Since $(x + 1)^2 > 0$ for $x \neq -1$, the sign of $f'(x)$ is determined by the numerator.

Answer

The function is increasing on

$$(-\infty, -1 - \sqrt{5}) \cup (-1 + \sqrt{5}, \infty).$$

The function is decreasing on

$$(-1 - \sqrt{5}, -1) \cup (-1, -1 + \sqrt{5}).$$

(b) Local extrema

At $x = -1 - \sqrt{5}$, the derivative changes from positive to negative, so there is a local maximum.

At $x = -1 + \sqrt{5}$, the derivative changes from negative to positive, so there is a local minimum.

$$f(-1 - \sqrt{5}) = -6 - 2\sqrt{5}, \quad f(-1 + \sqrt{5}) = -6 + 2\sqrt{5}.$$

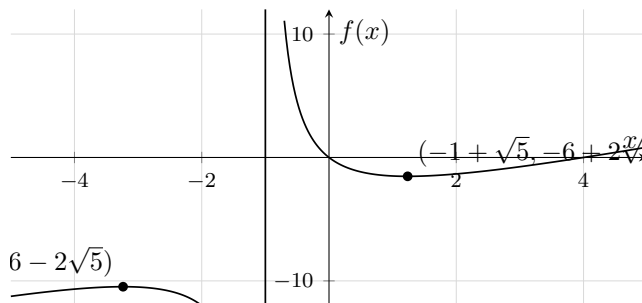
Answer

Local maximum at

$$(-1 - \sqrt{5}, -6 - 2\sqrt{5})$$

Local minimum at

$$(-1 + \sqrt{5}, -6 + 2\sqrt{5})$$

Graph of $f(x)$ 

Question 4**Problem:**

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum and minimum values.

Question 4 - Solution

We are given:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) Find the first derivative:

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$$

Test intervals for sign of $f'(x)$:

For $x < -1$: pick $x = -2$: $f'(-2) = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15 > 0$

For $-1 < x < 3$: pick $x = 0$: $f'(0) = -9 < 0$

For $x > 3$: pick $x = 4$: $f'(4) = 3(16) - 6(4) - 9 = 48 - 24 - 9 = 15 > 0$

Conclusion:

$f(x)$ is increasing on $(-\infty, -1) \cup (3, \infty)$

$f(x)$ is decreasing on $(-1, 3)$

(b) Local Extrema:

At $x = -1$: changes from increasing to decreasing $\rightarrow$ local maximum At $x = 3$: changes from decreasing to increasing $\rightarrow$ local minimum

Evaluate $f(x)$ at critical points:

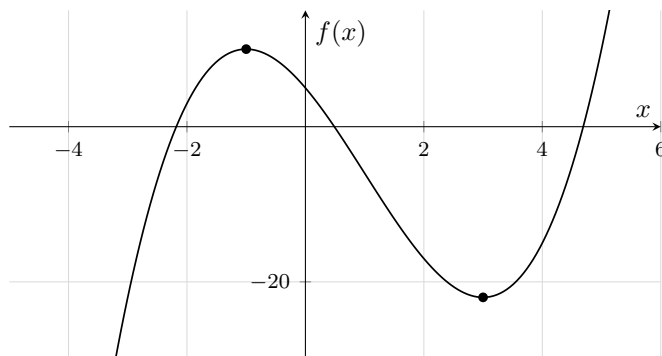
$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Local Maximum: $(-1, 10)$

Local Minimum: $(3, -22)$

Graph of $f(x) = x^3 - 3x^2 - 9x + 5$:



Question 5**Problem:**

Let

$$f(x) = x^4 - 4x^3 + 6$$

- (a) Find the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum and minimum values.

Question 5 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6$$

(a) **First Derivative:**

$$f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 4x^2(x - 3)$$

Set $f'(x) = 0$:

$$4x^2(x - 3) = 0 \Rightarrow x = 0, 3$$

Test intervals:

For $x < 0$: pick $x = -1$: $f'(-1) = 4(-1)^2(-1 - 3) = 4(1)(-4) = -16 < 0$

For $0 < x < 3$: pick $x = 1$: $f'(1) = 4(1)^2(1 - 3) = 4(-2) = -8 < 0$

For $x > 3$: pick $x = 4$: $f'(4) = 4(16)(1) = 64 > 0$

Conclusion:

$f(x)$ is decreasing on $(-\infty, 3)$

$f(x)$ is increasing on $(3, \infty)$

(b) **Local Extrema:**

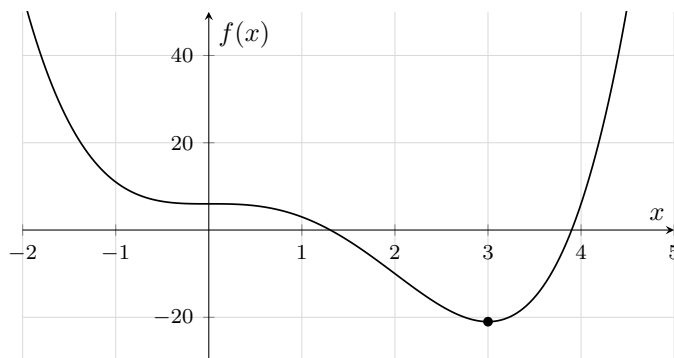
$f'(x)$ does not change sign around $x = 0$ (negative on both sides) $\rightarrow$ no extremum at $x = 0$ At $x = 3$:
from decreasing to increasing $\rightarrow$ local minimum

Evaluate:

$$f(3) = 3^4 - 4(3)^3 + 6 = 81 - 108 + 6 = -21$$

Local Minimum: $(3, -21)$

Graph of $f(x) = x^4 - 4x^3 + 6$:



Question 6**Problem:**

Let

$$f(x) = \frac{x^2}{x+1}$$

- (a) Find the critical points of $f(x)$ and determine where $f(x)$ is increasing or decreasing.
- (b) Identify any local maximum or minimum values.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^2}{x+1}$$

(a) First Derivative:

Use the quotient rule:

$$f'(x) = \frac{(2x)(x+1) - x^2(1)}{(x+1)^2} = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

Set $f'(x) = 0$:

$$\frac{x(x+2)}{(x+1)^2} = 0 \Rightarrow x = 0, -2$$

Also, note $f(x)$ is undefined at $x = -1$

Test intervals:

$(-\infty, -2)$: pick $x = -3$: numerator is positive, denominator is positive $\rightarrow f' > 0$

$(-2, -1)$: pick $x = -1.5$: numerator is negative, denominator is positive $\rightarrow f' < 0$

$(-1, 0)$: pick $x = -0.5$: numerator is negative, denominator is positive $\rightarrow f' < 0$

$(0, \infty)$: pick $x = 1$: numerator is positive, denominator is positive $\rightarrow f' > 0$

Conclusion:

$f(x)$ increases on $(-\infty, -2) \cup (0, \infty)$

$f(x)$ decreases on $(-2, -1) \cup (-1, 0)$

(b) Local Extrema:

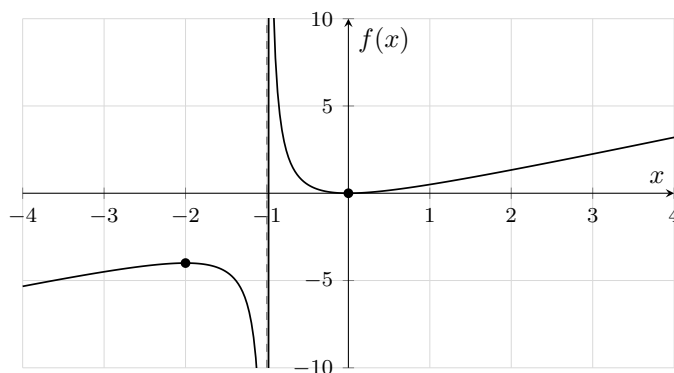
At $x = -2$: from increasing to decreasing $\rightarrow$ local max $f(-2) = \frac{(-2)^2}{-2+1} = \frac{4}{-1} = -4$

At $x = 0$: from decreasing to increasing $\rightarrow$ local min $f(0) = 0$

Answer:

Local maximum at $(-2, -4)$, Local minimum at $(0, 0)$

Graph of $f(x) = \frac{x^2}{x+1}$:



Question 7**Problem:**

Let

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find the critical points of $f(x)$ and determine the intervals where $f(x)$ is increasing or decreasing.
- (b) Identify all local maximum and minimum values.

Question 7 - Solution

We are given:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

(a) **First Derivative:**

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$$

Test Intervals:

$(-\infty, -1)$: pick $x = -2$: $f'(-2) = 3(-2)^2 - 6(-2) - 9 = 12 + 12 - 9 = 15 > 0$

$(-1, 3)$: pick $x = 0$: $f'(0) = -9 < 0$

$(3, \infty)$: pick $x = 4$: $f'(4) = 3(16) - 6(4) - 9 = 48 - 24 - 9 = 15 > 0$

Conclusion:

$f(x)$ is increasing on $(-\infty, -1) \cup (3, \infty)$

$f(x)$ is decreasing on $(-1, 3)$

(b) **Local Extrema:**

At $x = -1$: changes from increasing to decreasing $\rightarrow$ local maximum

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

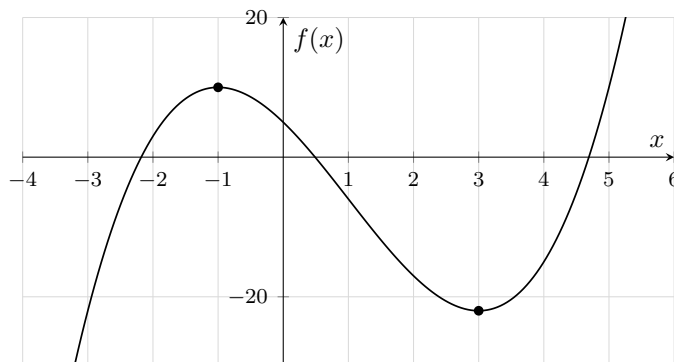
At $x = 3$: changes from decreasing to increasing $\rightarrow$ local minimum

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Answer:

Local maximum at $(-1, 10)$, Local minimum at $(3, -22)$

Graph of $f(x) = x^3 - 3x^2 - 9x + 5$:



Question 8**Problem:**

Let

$$f(x) = \frac{x^2}{x+2}$$

- (a) Find the domain of $f(x)$ and determine all critical points.
- (b) Identify intervals where $f(x)$ is increasing or decreasing.
- (c) Classify each critical point as a local maximum, local minimum, or neither.

Question 8 - Solution

We are given:

$$f(x) = \frac{x^2}{x+2}$$

(a) Domain and Critical Points:

The function is undefined at $x = -2$, so the domain is:

$$(-\infty, -2) \cup (-2, \infty)$$

Find the derivative using the quotient rule:

$$f'(x) = \frac{(2x)(x+2) - x^2(1)}{(x+2)^2} = \frac{2x(x+2) - x^2}{(x+2)^2} = \frac{2x^2 + 4x - x^2}{(x+2)^2} = \frac{x^2 + 4x}{(x+2)^2}$$

Set the derivative to zero:

$$f'(x) = 0 \Rightarrow x^2 + 4x = 0 \Rightarrow x(x+4) = 0 \Rightarrow x = 0, x = -4$$

Critical Points: $x = -4$ and $x = 0$

(b) Test Intervals for Increasing/Decreasing:

Test sign of $f'(x) = \frac{x(x+4)}{(x+2)^2}$

Interval $(-\infty, -4)$: pick $x = -5$: $f'(-5) = \frac{-5(-1)}{(-3)^2} = \frac{5}{9} > 0$

Interval $(-4, -2)$: pick $x = -3$: $f'(-3) = \frac{-3(1)}{(1)^2} = -3 < 0$

Interval $(-2, 0)$: pick $x = -1$: $f'(-1) = \frac{-1(3)}{(1)^2} = -3 < 0$

Interval $(0, \infty)$: pick $x = 1$: $f'(1) = \frac{1(5)}{(3)^2} = \frac{5}{9} > 0$

Increasing: $(-\infty, -4) \cup (0, \infty)$ **Decreasing:** $(-4, -2) \cup (-2, 0)$

(c) Classify Critical Points:

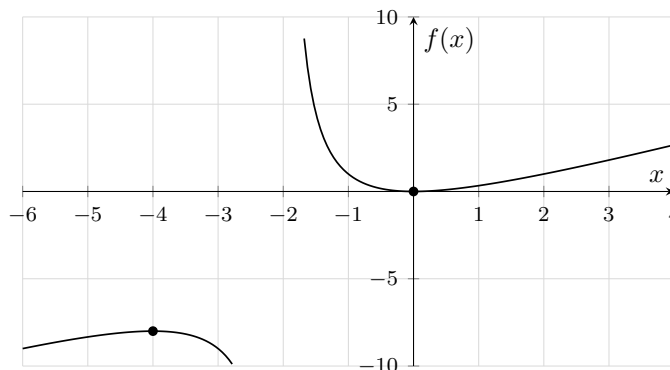
At $x = -4$: changes from increasing to decreasing $\rightarrow$ local max $f(-4) = \frac{16}{-2} = -8$

At $x = 0$: changes from decreasing to increasing $\rightarrow$ local min $f(0) = 0$

Conclusion:

Local maximum at $(-4, -8)$, Local minimum at $(0, 0)$

Graph of $f(x) = \frac{x^2}{x+2}$:



Question 9**Problem:**

Let

$$f(x) = x^3 - 6x^2 + 9x + 1$$

- (a) Find all critical points of $f(x)$.
- (b) Determine the intervals where $f(x)$ is increasing or decreasing.
- (c) Identify and classify all local extrema.

Question 9 - Solution

We are given:

$$f(x) = x^3 - 6x^2 + 9x + 1$$

(a) Critical Points:

Compute the first derivative:

$$f'(x) = 3x^2 - 12x + 9$$

Set derivative equal to 0:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1, x = 3$$

Critical points: $x = 1, x = 3$

(b) Increasing/Decreasing Intervals:

Test sign of $f'(x)$ in each interval:

$(-\infty, 1)$: pick $x = 0$, $f'(0) = 9 > 0$

$(1, 3)$: pick $x = 2$, $f'(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3 < 0$

$(3, \infty)$: pick $x = 4$, $f'(4) = 3(16) - 48 + 9 = 48 - 48 + 9 = 9 > 0$

Increasing: $(-\infty, 1) \cup (3, \infty)$ **Decreasing:** $(1, 3)$

(c) Local Extrema:

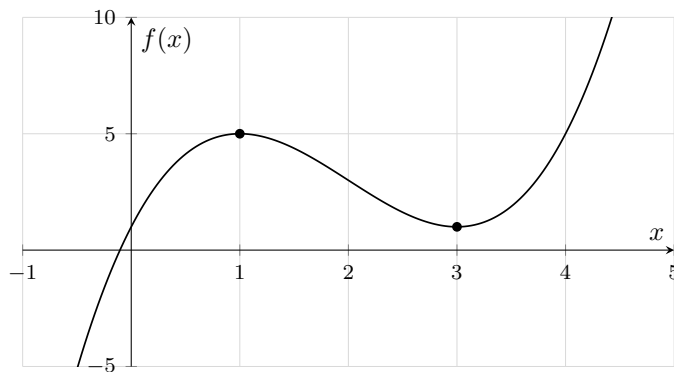
At $x = 1$: increasing to decreasing $\rightarrow$ local max $f(1) = 1 - 6 + 9 + 1 = 5$

At $x = 3$: decreasing to increasing $\rightarrow$ local min $f(3) = 27 - 54 + 27 + 1 = 1$

Conclusion:

Local maximum at $(1, 5)$, Local minimum at $(3, 1)$

Graph of $f(x) = x^3 - 6x^2 + 9x + 1$:



Question 10**Problem:**

Let

$$f(x) = \frac{x}{x^2 + 1}$$

- (a) Find the critical points of $f(x)$.
- (b) Determine where the function is increasing and decreasing.
- (c) Identify and classify any local extrema.

Question 10 - Solution

We are given:

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Critical Points:

Use the quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Set numerator equal to zero:

$$-x^2 + 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Critical points: $x = -1, x = 1$

(b) Increasing/Decreasing Intervals:

Analyze the sign of $f'(x) = \frac{1-x^2}{(x^2+1)^2}$

Denominator is always positive. So sign depends on numerator $1 - x^2$:

$x < -1$: $f'(x) < 0$ $-1 < x < 1$: $f'(x) > 0$ $x > 1$: $f'(x) < 0$

Increasing: $(-1, 1)$ **Decreasing:** $(-\infty, -1) \cup (1, \infty)$

(c) Local Extrema:

$f'(x)$ changes from negative to positive at $x = -1 \rightarrow$ local min $f(-1) = \frac{-1}{1+1} = -\frac{1}{2}$

$f'(x)$ changes from positive to negative at $x = 1 \rightarrow$ local max $f(1) = \frac{1}{1+1} = \frac{1}{2}$

Conclusion: Local minimum at $(-1, -\frac{1}{2})$, Local maximum at $(1, \frac{1}{2})$

Graph of $f(x) = \frac{x}{x^2+1}$:

