

Question 1

Let:

$$f(x) = 2x^3 - 9x^2 + 12x + 1$$

- (a) Find the critical points of $f(x)$.
- (b) Use the second derivative test to determine whether each critical point is a local minimum, maximum, or neither.

Question 1 - Solution

We are given:

$$f(x) = 2x^3 - 9x^2 + 12x + 1$$

(a) First derivative:

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$

Critical points: $x = 1, 2$

(b) Use the second derivative:

$$f''(x) = \frac{d}{dx}(f'(x)) = 12x - 18$$

Evaluate at each critical point:

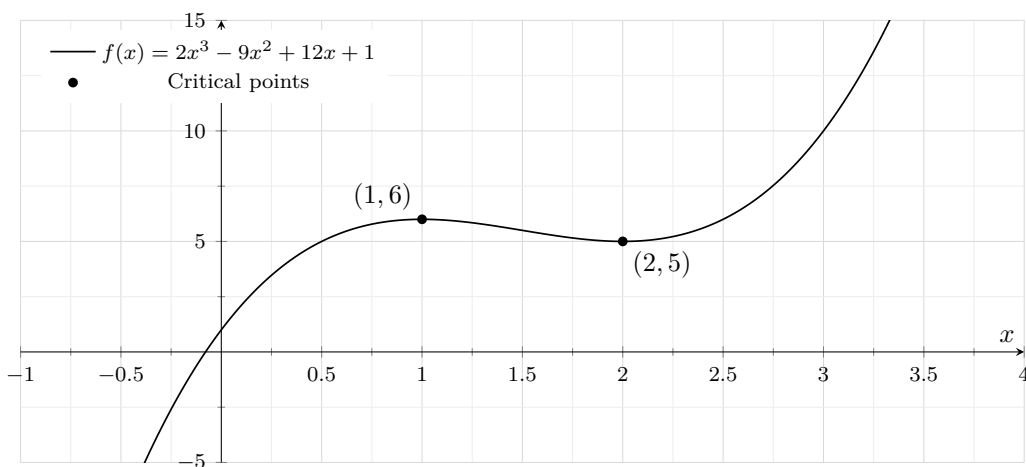
$$f''(1) = 12(1) - 18 = -6 \Rightarrow \text{local maximum at } x = 1$$

$$f''(2) = 12(2) - 18 = 6 \Rightarrow \text{local minimum at } x = 2$$

Answer:

Local maximum at $x = 1$

Local minimum at $x = 2$

Graph of $f(x)$ 

Question 2**Problem:**

Let $f(x) = x^3 - 6x^2 + 9x + 2$ on the closed interval $[0, 4]$.

- (a) Find the critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the interval $[0, 4]$.

Question 2 - Solution**(a) Find critical points:**

First, compute the derivative:

$$f'(x) = 3x^2 - 12x + 9$$

Set the derivative to zero:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 1)(x - 3) = 0 \Rightarrow x = 1, x = 3$$

Both critical points are in the interval $[0, 4]$.

(b) Evaluate function at critical points and endpoints:

$$f(0) = 0 - 0 + 0 + 2 = 2$$

$$f(1) = 1 - 6 + 9 + 2 = 6$$

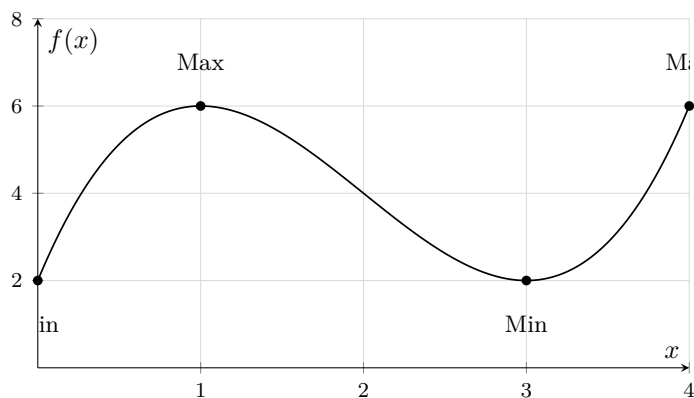
$$f(3) = 27 - 54 + 27 + 2 = 2$$

$$f(4) = 64 - 96 + 36 + 2 = 6$$

Conclusion:

- Maximum value: $\boxed{6}$ at $x = 1$ and $x = 4$ - Minimum value: $\boxed{2}$ at $x = 0$ and $x = 3$

Graph of the function on $[0, 4]$:



Question 3**Problem:**

Consider the function $f(x) = x^4 - 4x^3 + 10$ on the interval $[0, 4]$.

- (a) Find the critical points of $f(x)$ on the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the interval $[0, 4]$.

Question 3 - Solution**(a) Find critical points:**

First, compute the derivative:

$$f'(x) = 4x^3 - 12x^2$$

Factor:

$$f'(x) = 4x^2(x - 3) \Rightarrow f'(x) = 0 \text{ when } x = 0 \text{ or } x = 3$$

Both are within the interval $[0, 4]$, so check both.

(b) Evaluate function at critical points and endpoints:

$$f(0) = 0 - 0 + 10 = 10$$

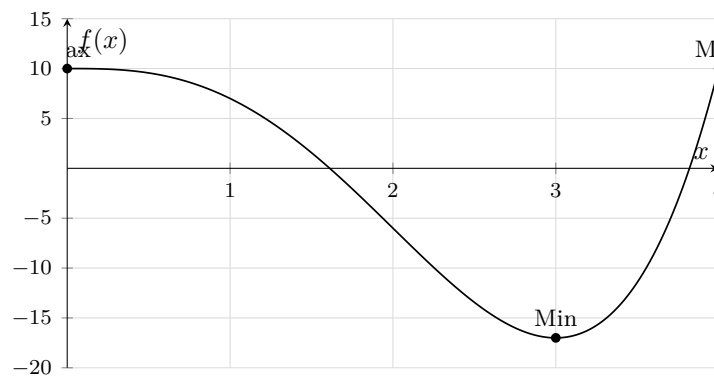
$$f(3) = 81 - 108 + 10 = -17$$

$$f(4) = 256 - 256 + 10 = 10$$

Conclusion:

- Minimum value: -17 at $x = 3$ - Maximum value: 10 at $x = 0$ and $x = 4$

Graph of the function on $[0, 4]$:



Question 4**Problem:**

Let $f(x) = \frac{x}{x^2+1}$, and consider the interval $[-2, 2]$.

- (a) Find the critical points of $f(x)$ on the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on $[-2, 2]$.

Question 4 - Solution**(a) Find the critical points:**

We find $f'(x)$ using the quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Set the numerator equal to zero:

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

Both $x = -1$ and $x = 1$ are in the interval $[-2, 2]$.

(b) Evaluate at critical points and endpoints:

$$f(-2) = \frac{-2}{4 + 1} = \frac{-2}{5}$$

$$f(-1) = \frac{-1}{1 + 1} = \frac{-1}{2}$$

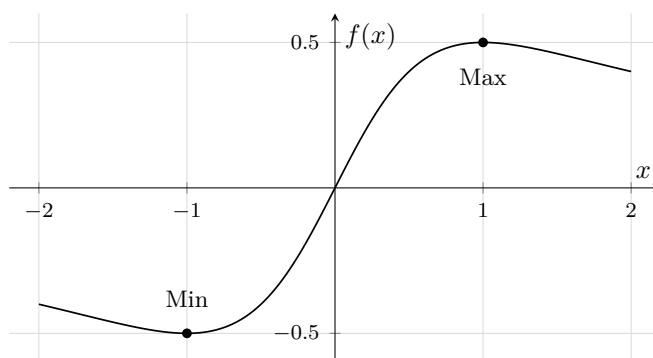
$$f(1) = \frac{1}{1 + 1} = \frac{1}{2}$$

$$f(2) = \frac{2}{4 + 1} = \frac{2}{5}$$

Conclusion:

- Maximum value: $\boxed{\frac{1}{2}}$ at $x = 1$ - Minimum value: $\boxed{-\frac{1}{2}}$ at $x = -1$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-2, 2]$:



Question 5**Problem:**

Let $f(x) = x^3 - 3x + 1$ on the interval $[-3, 3]$.

- (a) Find all critical points of $f(x)$ on the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on $[-3, 3]$.

Question 5 - Solution**(a) Find critical points:**

Take the derivative:

$$f'(x) = 3x^2 - 3$$

Set derivative equal to 0:

$$3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

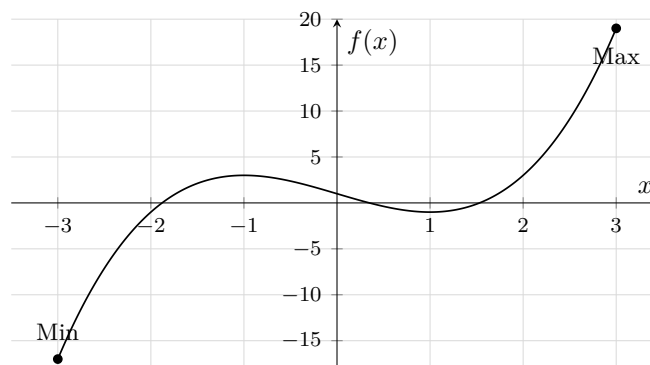
Critical points: $x = -1, x = 1$ **(b) Evaluate at critical points and endpoints:**

$$f(-3) = (-3)^3 - 3(-3) + 1 = -27 + 9 + 1 = -17$$

$$f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$$

$$f(1) = 1^3 - 3(1) + 1 = 1 - 3 + 1 = -1$$

$$f(3) = 27 - 9 + 1 = 19$$

Conclusion:- Absolute maximum: $\boxed{19}$ at $x = 3$ - Absolute minimum: $\boxed{-17}$ at $x = -3$ **Graph of** $f(x) = x^3 - 3x + 1$ on $[-3, 3]$:

Question 6**Problem:**

Consider the function $f(x) = x^4 - 4x^2$ on the interval $[-3, 3]$.

- (a) Find all critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on $[-3, 3]$.

Question 6 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 4x^3 - 8x$$

Set derivative to zero:

$$4x^3 - 8x = 0 \Rightarrow 4x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

Critical points in $[-3, 3]$: $x = -\sqrt{2}, 0, \sqrt{2}$ (b) Evaluate $f(x)$ at critical points and endpoints:

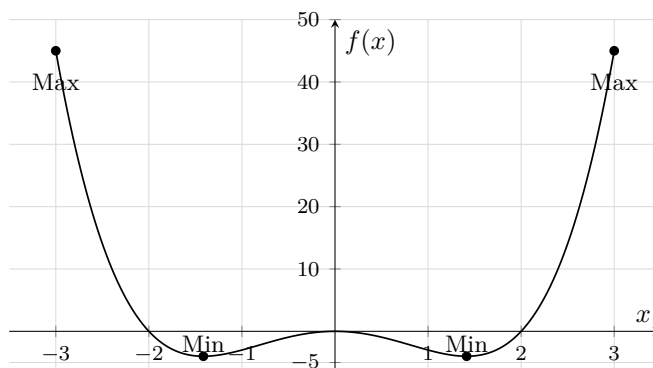
$$f(-3) = (-3)^4 - 4(-3)^2 = 81 - 36 = 45$$

$$f(-\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = 4 - 8 = -4$$

$$f(0) = 0^4 - 4(0)^2 = 0$$

$$f(\sqrt{2}) = 4 - 8 = -4$$

$$f(3) = 81 - 36 = 45$$

Conclusion:- Absolute maximum: $\boxed{45}$ at $x = -3$ and $x = 3$ - Absolute minimum: $\boxed{-4}$ at $x = \pm\sqrt{2}$ **Graph of** $f(x) = x^4 - 4x^2$ on $[-3, 3]$:

Question 7**Problem:**

Let $f(x) = x^3 - 3x^2 + 1$ on the interval $[0, 4]$.

- (a) Find all critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

Question 7 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Set $f'(x) = 0$ to find critical points:

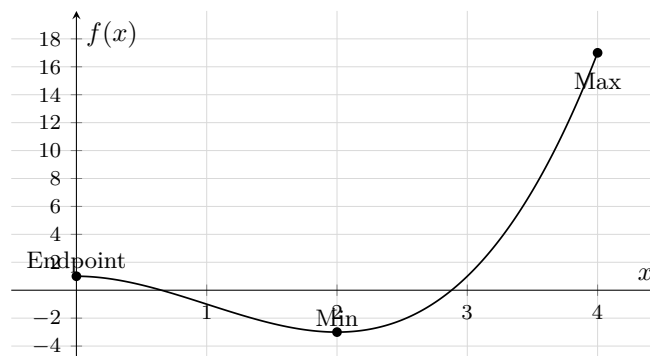
$$3x(x - 2) = 0 \Rightarrow x = 0, x = 2$$

(b) Evaluate function at critical points and endpoints:

$$f(0) = 0^3 - 3(0)^2 + 1 = 1$$

$$f(2) = 8 - 12 + 1 = -3$$

$$f(4) = 64 - 48 + 1 = 17$$

Conclusion:- Absolute maximum: $\boxed{17}$ at $x = 4$ - Absolute minimum: $\boxed{-3}$ at $x = 2$ **Graph of** $f(x) = x^3 - 3x^2 + 1$ on $[0, 4]$:

Question 8**Problem:**

Let $f(x) = \frac{x}{x^2+1}$ on the interval $[-3, 3]$.

- (a) Find all critical points of $f(x)$ in the interval.
- (b) Determine the absolute maximum and minimum values of $f(x)$ on the given interval.

Question 8 - Solution

(a) Find critical points:

Differentiate using the quotient rule:

$$f(x) = \frac{x}{x^2 + 1}, \quad f'(x) = \frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Set $f'(x) = 0$:

$$1 - x^2 = 0 \Rightarrow x = \pm 1$$

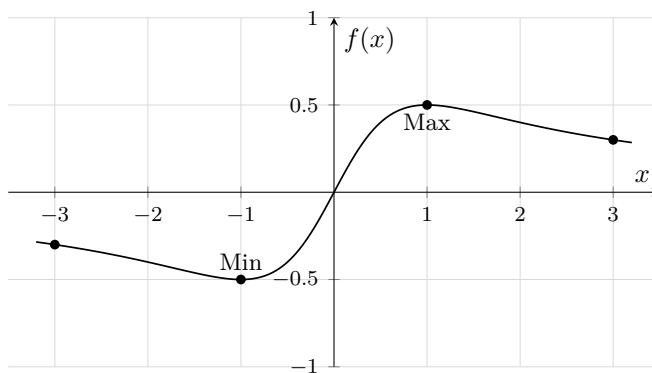
(b) Evaluate function at endpoints and critical points:

$$f(-3) = \frac{-3}{9+1} = -\frac{3}{10} \quad f(-1) = \frac{-1}{2} = -\frac{1}{2} \quad f(1) = \frac{1}{2} \quad f(3) = \frac{3}{10}$$

Conclusion:

- Absolute maximum: $\boxed{\frac{1}{2}}$ at $x = 1$ - Absolute minimum: $\boxed{-\frac{1}{2}}$ at $x = -1$

Graph of $f(x) = \frac{x}{x^2+1}$ on $[-3, 3]$:



Question 9**Problem:**

A farmer has 240 meters of fencing to enclose a rectangular field. One side of the rectangle lies along a river and does not require fencing.

- (a) Express the area of the field as a function of one variable.
- (b) Find the dimensions of the field that maximize the area.
- (c) What is the maximum area?

Question 9 - Solution

(a) Express the area as a function:

Let x be the length of the side perpendicular to the river. Since only three sides are fenced (two x 's and one along the length), the total fencing is:

$$2x + y = 240 \Rightarrow y = 240 - 2x$$

Area:

$$A(x) = x \cdot y = x(240 - 2x) = 240x - 2x^2$$

(b) Maximize the area:

Differentiate:

$$A'(x) = 240 - 4x$$

Set derivative to zero:

$$240 - 4x = 0 \Rightarrow x = 60$$

Check that this gives a maximum using second derivative:

$$A''(x) = -4 < 0 \Rightarrow \text{Maximum}$$

Find y :

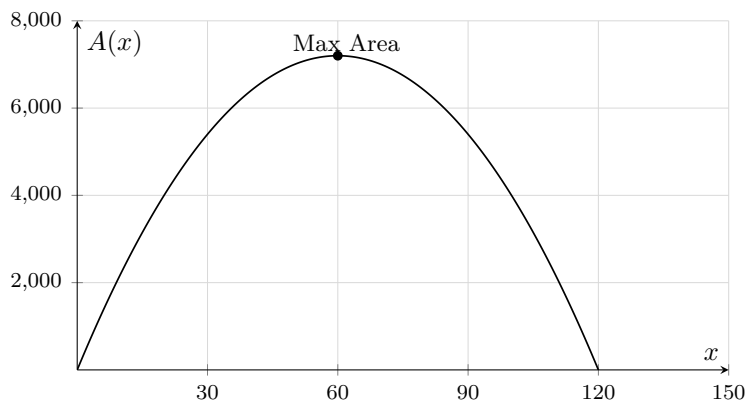
$$y = 240 - 2(60) = 120$$

Dimensions: 60 m $\times$ 120 m

(c) Maximum area:

$$A = 60 \cdot 120 = \text{7200 m}^2$$

Graph of $A(x) = 240x - 2x^2$:



Question 10**Problem:**

A box with a square base and open top must have a volume of 500 cm^3 . Find the dimensions of the box that minimize the surface area.

- (a) Express the surface area as a function of one variable.
- (b) Find the dimensions that minimize the surface area.
- (c) What is the minimum surface area?

Question 10 - Solution

Let x be the side length of the square base, and h be the height of the box.

(a) Surface area as a function of one variable:

Volume constraint:

$$x^2 h = 500 \Rightarrow h = \frac{500}{x^2}$$

Surface area (no top):

$$A(x) = x^2 + 4xh = x^2 + 4x \cdot \frac{500}{x^2} = x^2 + \frac{2000}{x}$$

(b) Minimize the surface area:

Differentiate:

$$A'(x) = 2x - \frac{2000}{x^2}$$

Set derivative to 0:

$$2x - \frac{2000}{x^2} = 0 \Rightarrow 2x^3 = 2000 \Rightarrow x^3 = 1000 \Rightarrow x = 10$$

Find h :

$$h = \frac{500}{10^2} = 5$$

Dimensions:

$$x = 10 \text{ cm}, h = 5 \text{ cm}$$

(c) Minimum surface area:

$$A = 10^2 + \frac{2000}{10} = 100 + 200 = \boxed{300 \text{ cm}^2}$$

Graph of $A(x) = x^2 + \frac{2000}{x}$:

