

Question 1

Let

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

- (a) Find all critical points of $f(x)$.
- (b) Classify each critical point as a local minimum, local maximum, or neither using the First Derivative Test.

Question 1 - Solution

We are given

$$f(x) = x^3 - 6x^2 + 9x + 1.$$

(a) Find $f'(x)$:

$$f'(x) = 3x^2 - 12x + 9.$$

Set $f'(x) = 0$ to find critical points:

$$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 1)(x - 3) = 0.$$

Critical points: $x = 1, 3$

(b) First Derivative Test:

- Choose test values around $x = 1$ and $x = 3$.
- Use $f'(x) = 3x^2 - 12x + 9$.

Evaluate the sign of $f'(x)$:

- On $(-\infty, 1)$, pick $x = 0$:

$$f'(0) = 9 > 0.$$

- On $(1, 3)$, pick $x = 2$:

$$f'(2) = 3(4) - 24 + 9 = -3 < 0.$$

- On $(3, \infty)$, pick $x = 4$:

$$f'(4) = 48 - 48 + 9 = 9 > 0.$$

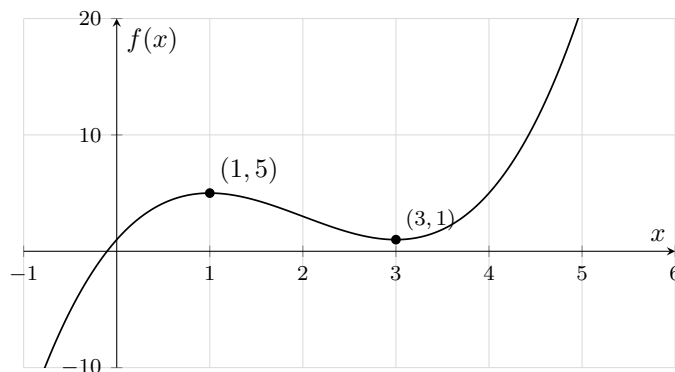
Conclusion:

- At $x = 1$, $f'(x)$ changes from positive to negative, so f has a local maximum.
- At $x = 3$, $f'(x)$ changes from negative to positive, so f has a local minimum.

Answer:

- Local maximum at $x = 1$ with $f(1) = 5$
- Local minimum at $x = 3$ with $f(3) = 1$

Graph of $f(x)$ with critical points marked



Question 2

A company models its profit $P(x)$, in thousands of dollars, as a function of the number of units x (in hundreds) it produces and sells:

$$P(x) = -2x^3 + 15x^2 + 36x.$$

- (a) Find the critical points of the function.
- (b) Use the First Derivative Test to classify each critical point as a local maximum, minimum, or neither.
- (c) How many units should be produced to maximize profit?

Question 2 - Solution

We are given

$$P(x) = -2x^3 + 15x^2 + 36x.$$

(a) Find $P'(x)$:

$$P'(x) = -6x^2 + 30x + 36.$$

Set $P'(x) = 0$ to find critical points:

$$-6x^2 + 30x + 36 = 0 \Rightarrow x^2 - 5x - 6 = 0 \Rightarrow (x - 6)(x + 1) = 0.$$

Critical points: $x = -1, 6$

(b) First Derivative Test:

Evaluate the sign of $P'(x)$ on intervals determined by the critical points.

- On $(-\infty, -1)$, pick $x = -2$:

$$P'(-2) = -6(4) + 30(-2) + 36 = -24 - 60 + 36 = -48 < 0.$$

- On $(-1, 6)$, pick $x = 0$:

$$P'(0) = 36 > 0.$$

- On $(6, \infty)$, pick $x = 7$:

$$P'(7) = -6(49) + 30(7) + 36 = -294 + 210 + 36 = -48 < 0.$$

Conclusion:

- At $x = -1$, $P'(x)$ changes from negative to positive, so P has a local minimum.
- At $x = 6$, $P'(x)$ changes from positive to negative, so P has a local maximum.

(c) Maximizing profit:

Since $x = 6$ is a local maximum, the profit is maximized at $x = 6$ (hundreds of units), which is

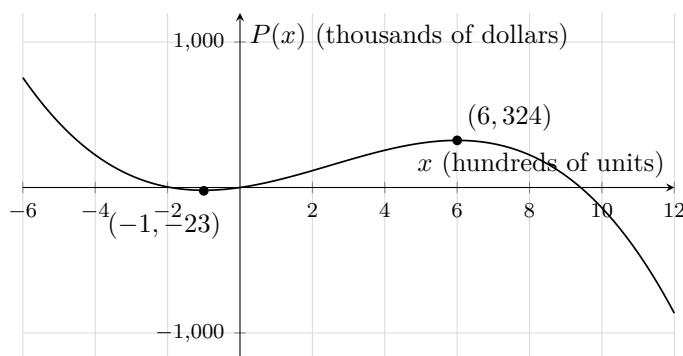
$$6 \times 100 = \boxed{600 \text{ units}}.$$

Compute the maximum profit:

$$P(6) = -2(6)^3 + 15(6)^2 + 36(6) = -432 + 540 + 216 = \boxed{324}.$$

This represents $\boxed{\$324,000}$ since P is in thousands of dollars.

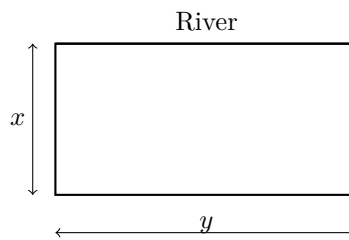
Graph of $P(x)$ with critical points marked



Question 3**Problem:**

A rectangular garden is to be built along a straight river. No fence is needed along the river, but fencing is required on the other three sides. If 120 meters of fencing is available, find the dimensions of the garden that maximize the enclosed area.

- (a) Define the objective function and constraint.
- (b) Use calculus to find the critical point(s).
- (c) Determine the dimensions that maximize the area.



Question 3 - Solution

Let x be the length perpendicular to the river, and y the length parallel to the river.

(a) Constraint:

Only three sides need fencing (two x 's and one y):

$$2x + y = 120 \quad \Rightarrow \quad y = 120 - 2x$$

Objective: Maximize the area:

$$A(x) = x \cdot y = x(120 - 2x) = 120x - 2x^2$$

(b) Find critical points:

Take the derivative:

$$A'(x) = 120 - 4x$$

Set $A'(x) = 0$:

$$120 - 4x = 0 \Rightarrow x = 30$$

(c) Check dimensions:

$$y = 120 - 2(30) = 60$$

Answer: The garden has dimensions $\boxed{30 \text{ m} \times 60 \text{ m}}$, giving a maximum area.

Question 4**Problem**

A company models its revenue $R(x)$, in thousands of dollars, as a function of the number of items x (in hundreds) it produces and sells:

$$R(x) = -x^3 + 12x^2 + 9x.$$

- (a) Find all critical points of $R(x)$.
- (b) Use the First Derivative Test to classify each critical point as a local maximum, local minimum, or neither.
- (c) How many items should be produced to maximize revenue?

Question 4 - Solution

We are given

$$R(x) = -x^3 + 12x^2 + 9x.$$

(a) Find the derivative:

$$R'(x) = -3x^2 + 24x + 9.$$

Set $R'(x) = 0$:

$$-3x^2 + 24x + 9 = 0 \Rightarrow x^2 - 8x - 3 = 0.$$

Using the quadratic formula:

$$x = \frac{8 \pm \sqrt{64 + 12}}{2} = \frac{8 \pm \sqrt{76}}{2} = 4 \pm \sqrt{19}.$$

Critical points: $x = 4 - \sqrt{19}$, $4 + \sqrt{19}$

(b) First Derivative Test

Since $R'(x)$ is a downward-opening parabola, it is negative outside its roots and positive between them.

- At $x = 4 - \sqrt{19}$, $R'(x)$ changes from negative to positive, so $R(x)$ has a local minimum.
- At $x = 4 + \sqrt{19}$, $R'(x)$ changes from positive to negative, so $R(x)$ has a local maximum.

(c) Maximizing revenue

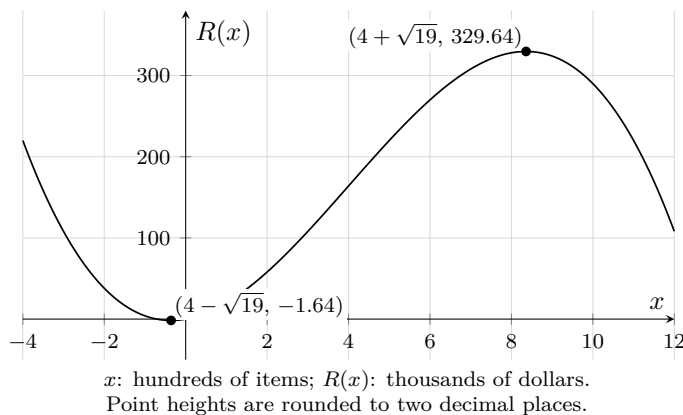
Revenue is maximized at

$$x = 4 + \sqrt{19} \approx 8.36.$$

Since x is measured in hundreds of items, this corresponds to

836 items.

Graph of $R(x)$ with critical points marked



Question 5**Problem**

Let

$$f(x) = x^4 - 4x^3 + 6x^2.$$

- (a) Find the critical points of $f(x)$.
- (b) Classify each critical point as a local maximum, local minimum, or neither.

Question 5 - Solution

We are given

$$f(x) = x^4 - 4x^3 + 6x^2.$$

(a) Find the critical points.

Differentiate:

$$f'(x) = 4x^3 - 12x^2 + 12x.$$

Factor:

$$f'(x) = 4x(x^2 - 3x + 3).$$

Set $f'(x) = 0$:

$$4x(x^2 - 3x + 3) = 0.$$

This gives $x = 0$, or $x^2 - 3x + 3 = 0$. The quadratic has discriminant

$$(-3)^2 - 4(1)(3) = 9 - 12 = -3 < 0,$$

so it has no real solutions. Therefore,

Only critical point: $x = 0$.

(b) Classify the critical point.

Compute the second derivative:

$$f''(x) = 12x^2 - 24x + 12.$$

Evaluate at $x = 0$:

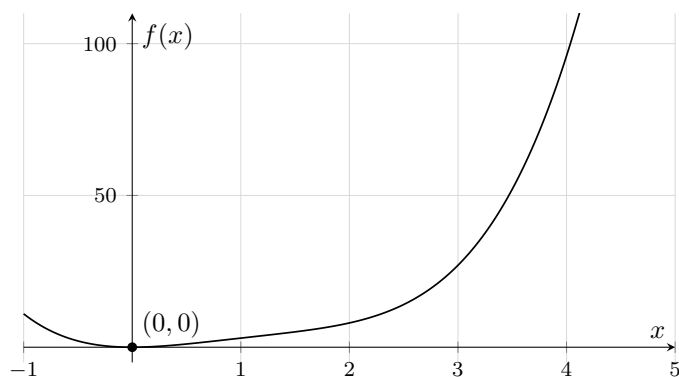
$$f''(0) = 12 > 0,$$

so $f(x)$ has a local minimum at $x = 0$. Also $f(0) = 0$, so the minimum point is $(0, 0)$.

Conclusion

Local minimum at $x = 0$ (point $(0, 0)$)

Graph of $f(x)$ with the critical point marked



Question 6**Problem:**Let $f(x) = \frac{x^3}{3} - 2x^2 + x + 5$.

- (a) Find all critical points of $f(x)$.
- (b) Use the First Derivative Test to classify each critical point as a local maximum, local minimum, or neither.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^3}{3} - 2x^2 + x + 5$$

(a) **Find the critical points:**

Take the first derivative:

$$f'(x) = x^2 - 4x + 1$$

Set $f'(x) = 0$:

$$x^2 - 4x + 1 = 0 \Rightarrow x = 2 \pm \sqrt{3}$$

Critical points: $x = 2 + \sqrt{3}$, $x = 2 - \sqrt{3}$

(b) **First Derivative Test:**

Evaluate $f'(x)$ on intervals around the critical points:

Interval $(-\infty, 2 - \sqrt{3})$: pick $x = 0$, $f'(0) = 1 > 0$

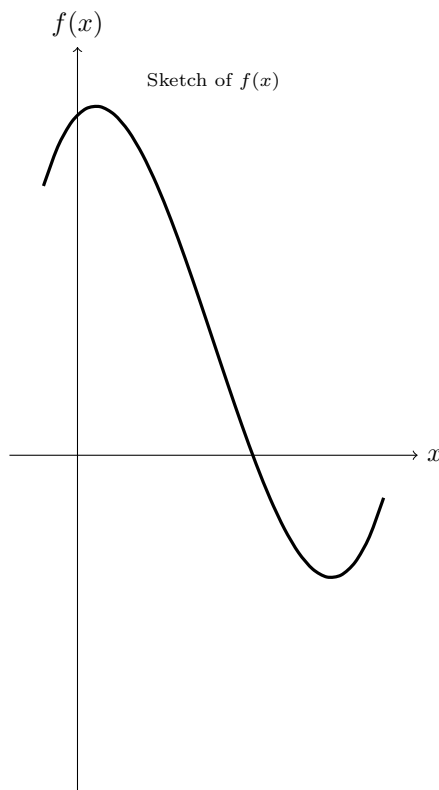
Interval $(2 - \sqrt{3}, 2 + \sqrt{3})$: pick $x = 2$, $f'(2) = -3 < 0$

Interval $(2 + \sqrt{3}, \infty)$: pick $x = 4$, $f'(4) = 1 > 0$

Conclusion:

$f'(x)$ changes from positive to negative at $x = 2 - \sqrt{3} \rightarrow$ local maximum
 $f'(x)$ changes from negative to positive at $x = 2 + \sqrt{3} \rightarrow$ local minimum

Local max at $x = 2 - \sqrt{3}$
Local min at $x = 2 + \sqrt{3}$



Question 7**Problem:**

A company's profit function (in thousands of dollars) from selling x hundred units of a product is given by:

$$P(x) = -x^3 + 6x^2 + 60$$

- (a) Find all critical points of $P(x)$.
- (b) Determine whether each critical point is a local maximum or minimum using the First Derivative Test.
- (c) Interpret your result in the context of the company's production.

Question 7 - Solution**(a) Find critical points:**

Take the derivative:

$$P'(x) = -3x^2 + 12x$$

Set $P'(x) = 0$:

$$-3x^2 + 12x = 0 \Rightarrow x(-3x + 12) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

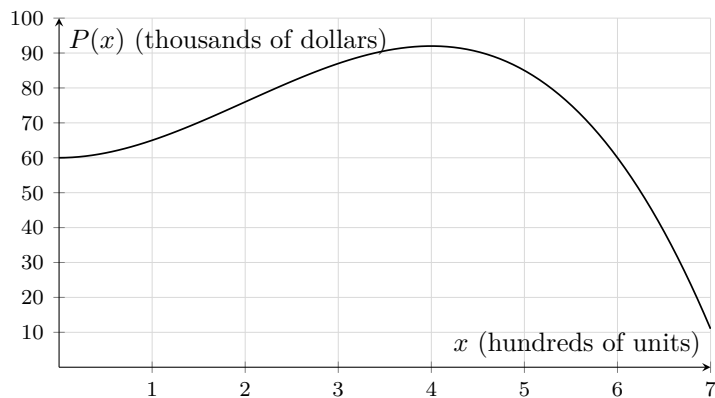
Critical points: $x = 0$, $x = 4$ **(b) First Derivative Test:**Check intervals: For $x < 0$: not relevant (negative production)Between $0 < x < 4$: pick $x = 2$, $P'(2) = -3(4) + 12(2) = -12 + 24 = 12 > 0$ For $x > 4$: pick $x = 5$, $P'(5) = -3(25) + 12(5) = -75 + 60 = -15 < 0$

Conclusion: $P(x)$ increases before $x = 4$, decreases after $\rightarrow$ local max at $x = 4$
 $x = 0$ is a stationary point, but no increase before it (since domain is $x \geq 0$)

(c) Interpretation:

Producing 400 units ($x = 4$) yields the maximum profit. Profit decreases beyond 400 units, and no benefit is gained producing less than 400.

Maximum profit occurs when the company produces 400 units.

Profit Function Graph:

Question 8**Problem:**

A ball is launched vertically upward, and its height in meters at time t seconds is modeled by:

$$h(t) = -5t^2 + 20t + 1$$

- (a) Find all critical points of the function $h(t)$.
- (b) Use the First Derivative Test to classify each critical point.
- (c) What is the maximum height reached by the ball and at what time?

Question 8 - Solution

(a) Find critical points:

Take the derivative:

$$h'(t) = -10t + 20$$

Set $h'(t) = 0$:

$$-10t + 20 = 0 \Rightarrow t = 2$$

Critical point: $t = 2$

(b) First Derivative Test:

For $t < 2$: pick $t = 1$, $h'(1) = -10(1) + 20 = 10 > 0$ For $t > 2$: pick $t = 3$, $h'(3) = -10(3) + 20 = -10 < 0$ Since $h'(t)$ changes from positive to negative at $t = 2$, this is a local maximum.

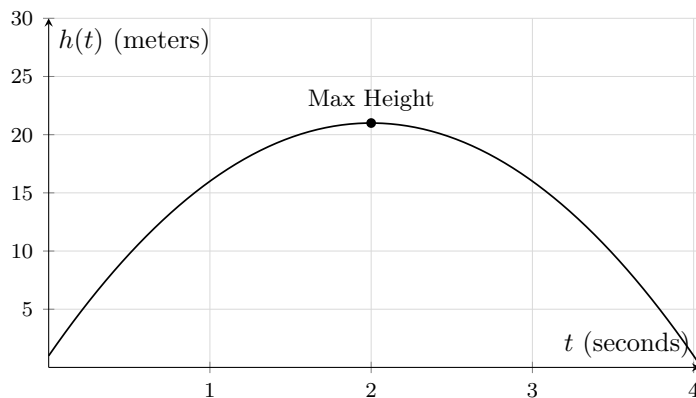
(c) Maximum Height:

Substitute $t = 2$ into $h(t)$:

$$h(2) = -5(4) + 20(2) + 1 = -20 + 40 + 1 = 21$$

Answer: The maximum height is 21 meters, reached at $t = 2$ seconds.

Graph of the Trajectory:



Question 9**Problem:**

Let the function be defined as:

$$f(x) = x^3 - 3x^2 - 9x + 5$$

- (a) Find all critical points of $f(x)$.
- (b) Use the First Derivative Test to classify each critical point as a local maximum or local minimum.
- (c) Determine the function value at each critical point.

Question 9 - Solution

(a) Find critical points:

Differentiate:

$$f'(x) = 3x^2 - 6x - 9$$

Set $f'(x) = 0$:

$$3x^2 - 6x - 9 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$$

Critical points: $x = -1, x = 3$

(b) First Derivative Test:

Interval $(-\infty, -1)$: Choose $x = -2$, $f'(-2) = 3(4) + 12 - 9 = 15 > 0$ Interval $(-1, 3)$: Choose $x = 0$, $f'(0) = -9 < 0$ Interval $(3, \infty)$: Choose $x = 4$, $f'(4) = 3(16) - 24 - 9 = 15 > 0$ So: At $x = -1$, f' changes from positive to negative $\rightarrow$ local maximum At $x = 3$, f' changes from negative to positive $\rightarrow$ local minimum

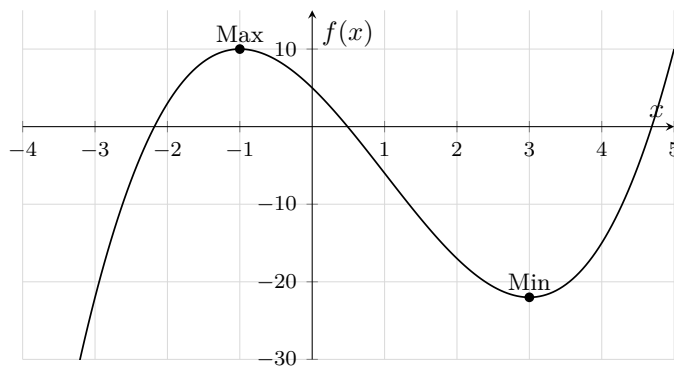
(c) Function values:

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -1 - 3 + 9 + 5 = 10$$

$$f(3) = 27 - 27 - 27 + 5 = -22$$

Local maximum: $f(-1) = 10$ Local minimum: $f(3) = -22$

Graph of the function:



Question 10**Problem:**

Consider the function:

$$f(x) = x^4 - 4x^3 + 6x^2 + 1$$

- (a) Find all critical points of $f(x)$.
- (b) Classify each critical point using the First Derivative Test.
- (c) Determine the function value at each critical point.

Question 10 - Solution**(a) Find critical points:**

First, compute the derivative:

$$f'(x) = 4x^3 - 12x^2 + 12x$$

Factor:

$$f'(x) = 4x(x^2 - 3x + 3)$$

Set derivative to zero:

$$4x(x^2 - 3x + 3) = 0 \Rightarrow x = 0 \quad (\text{real root})$$

Solve $x^2 - 3x + 3 = 0$:

$$\text{Discriminant: } D = (-3)^2 - 4(1)(3) = 9 - 12 = -3 \Rightarrow \text{no real roots}$$

Critical point: $x = 0$

(b) First Derivative Test: Test sign of $f'(x)$ around $x = 0$:

For $x < 0$: try $x = -1$

$$f'(-1) = 4(-1)^3 - 12(-1)^2 + 12(-1) = -4 - 12 - 12 = -28 < 0$$

For $x > 0$: try $x = 1$

$$f'(1) = 4(1)^3 - 12(1)^2 + 12(1) = 4 - 12 + 12 = 4 > 0$$

Since $f'(x)$ changes from negative to positive at $x = 0$, there is a local minimum.

(c) Function value at critical point:

$$f(0) = 0^4 - 4(0)^3 + 6(0)^2 + 1 = 1$$

Local minimum at: $(0, 1)$

Graph of the function:

