

**Question 1**

Use a linear approximation to estimate  $\sqrt{4.1}$ .

- (a) Define the function and the point where you linearize.
- (b) Find the linearization  $L(x)$  at the chosen point.
- (c) Use  $L(x)$  to estimate  $\sqrt{4.1}$ .

**Question 1 - Solution**

(a) Let  $f(x) = \sqrt{x}$

Choose point  $a = 4$ , since  $\sqrt{4} = 2$  is easy to compute.

(b) Linearization formula:

$$L(x) = f(a) + f'(a)(x - a)$$

Compute derivative:

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(4) = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

(c) Estimate  $\sqrt{4.1} \approx L(4.1)$ :

$$L(4.1) = 2 + \frac{1}{4}(0.1) = 2 + 0.025 = \boxed{2.025}$$

**Question 2**

Use a linear approximation to estimate the value of  $\sqrt[3]{8.1}$ .

- (a) Identify a function  $f(x)$ , a point  $a$ , and write the linear approximation  $L(x)$ .
- (b) Use  $L(x)$  to estimate  $\sqrt[3]{8.1}$ .
- (c) Compare the estimate to the actual value using a calculator.

**Question 2 - Solution**

Let  $f(x) = \sqrt[3]{x} = x^{1/3}$ . We choose  $a = 8$ , since  $\sqrt[3]{8} = 2$  is easy to compute.

**(a) Linear approximation formula:**

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f(x) = x^{1/3}, \quad f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(8) = \frac{1}{3(8)^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

So:

$$L(x) = f(8) + f'(8)(x - 8) = 2 + \frac{1}{12}(x - 8)$$

**(b) Estimate:**

$$L(8.1) = 2 + \frac{1}{12}(8.1 - 8) = 2 + \frac{1}{12}(0.1) = 2 + \frac{1}{120} \approx 2.0083$$

**(c) Actual value:**

$$\sqrt[3]{8.1} \approx 2.0083 \quad (\text{using calculator})$$

**Conclusion:** The linear approximation gives a very close estimate:

$\sqrt[3]{8.1} \approx 2.0083$

**Question 3**

Use a linear approximation to estimate the value of  $\ln(1.05)$ .

- (a) Identify a function  $f(x)$ , choose a point  $a$ , and write the linear approximation  $L(x)$ .
- (b) Use  $L(x)$  to estimate  $\ln(1.05)$ .
- (c) Compare the estimate to the actual value.

**Question 3 - Solution**

Let  $f(x) = \ln(x)$ , and choose  $a = 1$  because  $\ln(1) = 0$  is easy to compute.

**(a) Linear approximation formula:**

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

So:

$$L(x) = \ln(1) + 1 \cdot (x - 1) = 0 + (x - 1) = x - 1$$

**(b) Estimate:**

$$L(1.05) = 1.05 - 1 = 0.05$$

**(c) Actual value:**

$$\ln(1.05) \approx 0.04879 \quad (\text{using calculator})$$

**Conclusion:** The linear approximation gives:

$\ln(1.05) \approx 0.05$

which is a very good estimate.

**Question 4**

Use a linear approximation to estimate  $\sqrt[3]{8.1}$ .

- (a) Identify a function  $f(x)$ , choose a suitable point  $a$ , and write the linear approximation  $L(x)$ .
- (b) Use  $L(x)$  to estimate  $\sqrt[3]{8.1}$ .
- (c) Compare the estimate to the actual value.

**Question 4 - Solution**

Let  $f(x) = \sqrt[3]{x} = x^{1/3}$ , and choose  $a = 8$ , since  $\sqrt[3]{8} = 2$ .

(a) **Linear approximation formula:**

$$L(x) = f(a) + f'(a)(x - a)$$

Compute:

$$f'(x) = \frac{1}{3}x^{-2/3}, \quad f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

So:

$$L(x) = 2 + \frac{1}{12}(x - 8)$$

(b) **Estimate:**

$$L(8.1) = 2 + \frac{1}{12}(0.1) = 2 + \frac{1}{120} = \frac{241}{120} \approx 2.0083$$

(c) **Actual value:**

$$\sqrt[3]{8.1} \approx 2.00828 \quad (\text{using calculator})$$

**Conclusion:** The linear approximation gives:

$\sqrt[3]{8.1} \approx 2.0083$

which is very close to the actual value.

**Question 5**

Use linear approximation to estimate  $\ln(1.05)$ .

- (a) Define an appropriate function  $f(x)$ , and choose a convenient point  $a$  near 1.05.
- (b) Find the linear approximation  $L(x)$  of  $f(x)$  at  $a$ .
- (c) Use  $L(x)$  to approximate  $\ln(1.05)$ , and compare it with the actual value.

**Question 5 - Solution**

Let  $f(x) = \ln(x)$ . Choose  $a = 1$ , since  $\ln(1) = 0$  is simple and near 1.05.

(a) **Derivative of  $f(x)$ :**

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

(b) **Linear approximation:**

$$L(x) = f(a) + f'(a)(x - a) = \ln(1) + 1(x - 1) = x - 1$$

(c) **Estimate:**

$$L(1.05) = 1.05 - 1 = 0.05$$

Actual value:

$$\ln(1.05) \approx 0.04879 \quad (\text{using calculator})$$

**Conclusion:**

$$\boxed{\ln(1.05) \approx 0.05} \quad (\text{Linear Approximation})$$

Actual:  $\ln(1.05) \approx 0.04879 \Rightarrow$  Very close!

**Question 6**

Use linear approximation to estimate  $\sqrt[3]{8.2}$ .

- (a) Define an appropriate function  $f(x)$ , and choose a point  $a$  near 8.2.
- (b) Find the linear approximation  $L(x)$  of  $f(x)$  at  $a$ .
- (c) Use  $L(x)$  to estimate  $\sqrt[3]{8.2}$ , and compare it to the actual value.

**Question 6 - Solution**

Let  $f(x) = \sqrt[3]{x} = x^{1/3}$ , and choose  $a = 8$  since  $\sqrt[3]{8} = 2$  is exact and close to 8.2.

**(a) Compute derivative:**

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}, \quad \text{so} \quad f'(8) = \frac{1}{3\sqrt[3]{64}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

**(b) Linear approximation:**

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{12}(x - 8)$$

**(c) Estimate:**

$$L(8.2) = 2 + \frac{1}{12}(0.2) = 2 + \frac{0.2}{12} = 2 + \frac{1}{60} \approx 2.0167$$

Actual value:

$$\sqrt[3]{8.2} \approx 2.0164 \quad (\text{from calculator})$$

**Conclusion:**

$$\boxed{\sqrt[3]{8.2} \approx 2.0167} \quad (\text{Linear Approximation})$$

$$\text{Actual: } \sqrt[3]{8.2} \approx 2.0164 \Rightarrow \text{Very close!}$$

**Question 7**

Use linear approximation to estimate  $\ln(1.05)$ .

- (a) Define a function  $f(x)$  and choose a suitable point  $a$  near 1.05.
- (b) Find the linear approximation  $L(x)$  of  $f(x)$  at  $x = a$ .
- (c) Use  $L(x)$  to estimate  $\ln(1.05)$  and compare to the actual value.

**Question 7 - Solution**

Let  $f(x) = \ln(x)$ . We choose  $a = 1$  since  $\ln(1) = 0$  is easy to compute.

(a) Compute the derivative:

$$f'(x) = \frac{1}{x}, \quad \text{so } f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 0 + 1(x - 1) = x - 1$$

(c) Estimate:

$$L(1.05) = 1.05 - 1 = 0.05$$

**Compare to actual:**

$$\ln(1.05) \approx 0.04879 \quad (\text{from calculator})$$

**Conclusion:**

$$\boxed{\ln(1.05) \approx 0.05} \quad (\text{Linear Approximation})$$

Actual:  $\ln(1.05) \approx 0.04879 \Rightarrow$  Very close!

**Question 8**

Use a linear approximation to estimate  $\sqrt{10.2}$ .

- (a) Define an appropriate function  $f(x)$  and choose a point  $a$  close to 10.2 where  $f(a)$  is easy to compute.
- (b) Find the linear approximation  $L(x)$  to  $f(x)$  at  $x = a$ .
- (c) Use the approximation to estimate  $\sqrt{10.2}$ , and compare it with the actual value.

**Question 8 - Solution**

For  $f(x) = \sqrt{x}$ , use the nearby perfect square  $a = 9$ .

$$f(9) = 3, \quad f'(9) = \frac{1}{6}, \quad L(x) = 3 + \frac{x-9}{6}.$$

Therefore

$\sqrt{10.2} \approx L(10.2) = 3.2.$

Since  $f''(x) = -1/(4x^{3/2}) < 0$  for  $x > 0$ , the tangent line lies above the curve, so this is an overestimate. For comparison,  $\sqrt{10.2} \approx 3.193743885$ ; the error is approximately 0.006256115.

**Question 9**

Use a linear approximation to estimate  $\ln(1.05)$ .

- (a) Define a function  $f(x)$  such that  $f(x) = \ln(x)$ , and choose a value  $a$  near 1.05 where  $f(a)$  is easy to compute.
- (b) Find the linear approximation  $L(x)$  at  $x = a$ .
- (c) Use the approximation to estimate  $\ln(1.05)$  and compare with the actual value.

**Question 9 - Solution**

(a) Let  $f(x) = \ln(x)$ , and choose  $a = 1$  since  $\ln(1) = 0$  is easy to compute.

$$f(a) = \ln(1) = 0, \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

(b) Linear approximation:

$$L(x) = f(a) + f'(a)(x - a) = 0 + 1(x - 1) = x - 1$$

(c) Estimate  $\ln(1.05)$  using:

$$L(1.05) = 1.05 - 1 = 0.05$$

**Actual:**  $\ln(1.05) \approx 0.04879$

**Conclusion:**

$$\boxed{\ln(1.05) \approx 0.05} \quad (\text{Linear Approximation})$$

**Question 10**

Use a linear approximation to estimate  $\sqrt{10.2}$ .

- (a) Let  $f(x) = \sqrt{x}$ . Choose a value  $a$  near 10.2 where the square root is easy to compute.
- (b) Find the linear approximation  $L(x)$  at  $x = a$ .
- (c) Use the approximation to estimate  $\sqrt{10.2}$  and compare with the actual value.

**Question 10 - Solution**

For  $f(x) = \sqrt{x}$ , use the nearby perfect square  $a = 9$ .

$$f(9) = 3, \quad f'(9) = \frac{1}{6}, \quad L(x) = 3 + \frac{x-9}{6}.$$

Therefore

$$\boxed{\sqrt{10.2} \approx L(10.2) = 3.2.}$$

Since  $f''(x) = -1/(4x^{3/2}) < 0$  for  $x > 0$ , the tangent line lies above the curve, so this is an overestimate. For comparison,  $\sqrt{10.2} \approx 3.193743885$ ; the error is approximately 0.006256115.