

Question 1

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

- (a) Verify that the limit results in an indeterminate form.
- (b) Use L'Hospital's Rule to evaluate the limit.

Question 1 - Solution

(a) As $x \rightarrow 0$:

- $\sin(3x) \rightarrow 0$, and $x \rightarrow 0$ - So, the limit is of the indeterminate form $\frac{0}{0}$

(b) Apply L'Hospital's Rule:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\sin(3x)]}{\frac{d}{dx}[x]} \\ &= \lim_{x \rightarrow 0} \frac{3 \cos(3x)}{1} = 3 \cos(0) = 3(1) = \boxed{3}\end{aligned}$$

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

Question 2 - Solution

We observe that both $\frac{1}{x} \rightarrow \infty$ and $\frac{1}{\sin x} \rightarrow \infty$ as $x \rightarrow 0^+$, so the expression is of the indeterminate form:

$$\infty - \infty$$

We first combine into a single fraction:

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$$

Now the limit becomes:

$$\lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x}$$

As $x \rightarrow 0^+$, both numerator and denominator $\rightarrow 0$. This is now a $\frac{0}{0}$ indeterminate form, so we apply L'Hospital's Rule:

$$\lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$

Now evaluate the new limit:

- Numerator: $\cos x - 1 \rightarrow 0$ - Denominator: $\sin x + x \cos x \rightarrow 0$

Still indeterminate! Apply L'Hospital's Rule again:

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \rightarrow 0^+} \frac{-\sin x}{2 \cos x - x \sin x}$$

Now evaluate: - Numerator: $-\sin x \rightarrow 0$ - Denominator: $2 \cos x - x \sin x \rightarrow 2$

So the final result is:

$$\boxed{0}$$

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(x \ln \left(1 + \frac{1}{x} \right) \right)$$

Question 3 - Solution

We observe that as $x \rightarrow \infty$:

$$- \ln\left(1 + \frac{1}{x}\right) \rightarrow 0 \quad - x \rightarrow \infty$$

So the expression becomes:

$$\infty \cdot 0 \quad (\text{indeterminate form})$$

We rewrite it as a quotient:

$$x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x}$$

Now as $x \rightarrow \infty$, we get a $\frac{0}{0}$ form, so we can apply L'Hospital's Rule.

Let's compute the limit using L'Hospital:

$$\lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln\left(1 + \frac{1}{x}\right)}{\frac{d}{dx} \left(\frac{1}{x}\right)}$$

Differentiate numerator and denominator:

- Numerator derivative:

$$\frac{d}{dx} \ln\left(1 + \frac{1}{x}\right) = \frac{-1}{x^2\left(1 + \frac{1}{x}\right)} = \frac{-1}{x(x+1)}$$

- Denominator derivative:

$$\frac{d}{dx} \left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Now plug in:

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{-1/(x(x+1))}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{x^2}{x(x+1)} = \lim_{x \rightarrow \infty} \frac{x}{x+1} \\ &\lim_{x \rightarrow \infty} \frac{x}{x+1} = 1 \end{aligned}$$

Final Answer:

$$\boxed{1}$$

Question 4

Evaluate the limit:

$$\lim_{x \rightarrow 0^+} (1 + 2x)^{\frac{1}{x}}$$

Question 4 - Solution

We identify the limit as the indeterminate form:

$$\lim_{x \rightarrow 0^+} (1 + 2x)^{\frac{1}{x}} = 1^\infty$$

Let:

$$y = (1 + 2x)^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x} \ln(1 + 2x)$$

Now compute the limit:

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(1 + 2x)}{x}$$

This is a $\frac{0}{0}$ form. Apply L'Hospital's Rule:

$$= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} \ln(1 + 2x)}{\frac{d}{dx} x} = \lim_{x \rightarrow 0^+} \frac{\frac{2}{1+2x}}{1} = \frac{2}{1} = 2$$

So:

$$\ln y \rightarrow 2 \quad \Rightarrow \quad y \rightarrow e^2$$

Final Answer:

$$\boxed{e^2}$$

Question 5

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$$

Question 5 - Solution

As $x \rightarrow \infty$, both numerator and denominator approach infinity:

$$\frac{\ln x}{\sqrt{x}} \rightarrow \frac{\infty}{\infty} \quad (\text{indeterminate form})$$

Apply L'Hospital's Rule:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} \sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

Final Answer:

$$\boxed{0}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow \infty} (x - \ln x)$$

Question 6 - Solution

As $x \rightarrow \infty$, we observe:

$$x \rightarrow \infty, \quad \ln x \rightarrow \infty$$

So the limit is of the form:

$$\infty - \infty \quad (\text{indeterminate form})$$

To apply L'Hospital's Rule, we rewrite the expression to form a quotient. Factor:

$$x - \ln x = x \left(1 - \frac{\ln x}{x} \right)$$

Now evaluate:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \quad (\text{since exponential grows faster})$$

So:

$$\lim_{x \rightarrow \infty} (x - \ln x) = \infty \cdot (1 - 0) = \infty$$

Final Answer: ∞

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

Question 7 - Solution

As $x \rightarrow 0$, we know:

$$\tan x \rightarrow 0, \quad x \rightarrow 0 \Rightarrow \tan x - x \rightarrow 0 \quad \text{and} \quad x^3 \rightarrow 0 \Rightarrow \text{Indeterminate form } \frac{0}{0}$$

Apply L'Hospital's Rule:

First derivative:

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\tan x - x]}{\frac{d}{dx}[x^3]} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \rightarrow 0$, $\sec^2 x \rightarrow 1$, so numerator $\rightarrow 0$, denominator $\rightarrow 0$, still $\frac{0}{0}$

Second derivative:

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\sec^2 x - 1]}{\frac{d}{dx}[3x^2]} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \cdot \sec x \tan x}{6x}$$

As $x \rightarrow 0$, $\tan x \rightarrow 0$, so numerator $\rightarrow 0$, denominator $\rightarrow 0$

Third derivative:

Differentiate numerator and denominator again. Let's simplify instead:

Use the series:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \Rightarrow \tan x - x = \frac{x^3}{3} + \cdots$$

Then:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + \cdots}{x^3} = \frac{1}{3}$$

Final Answer:

$$\boxed{\frac{1}{3}}$$

Question 8

Evaluate the limit:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$$

Question 8 - Solution

As $x \rightarrow \infty$, we have:

$$\sqrt{x^2 + x} \rightarrow \infty, \quad x \rightarrow \infty \Rightarrow \text{Indeterminate form: } \infty - \infty$$

To resolve this, multiply by the conjugate:

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x) &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + x)}{\sqrt{x^2 + x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + x) - x^2}{\sqrt{x^2 + x} + x} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + x} + x} \end{aligned}$$

Now simplify numerator and denominator:

Factor x from the square root:

$$= \lim_{x \rightarrow \infty} \frac{x}{x \left(\sqrt{1 + \frac{1}{x}} + 1 \right)} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1}$$

As $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0$, so:

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

Final Answer:

$$\boxed{\frac{1}{2}}$$

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

Question 9 - Solution

As $x \rightarrow 0$, we have:

$$\tan x \rightarrow 0, \quad x \rightarrow 0 \quad \Rightarrow \quad \frac{\tan x - x}{x^3} \rightarrow \frac{0}{0} \quad (\text{indeterminate form})$$

Apply L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \rightarrow 0$, $\sec^2 x \rightarrow 1$, so the numerator goes to 0, denominator to 0:

$$\Rightarrow \frac{0}{0} \quad \text{Apply L'Hospital's Rule again}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} = \lim_{x \rightarrow 0} \frac{2 \cdot 1 \cdot 0}{6 \cdot 0} = \frac{0}{0} \quad (\text{Apply once more})$$

Take third derivative:

Numerator derivative:

$$\frac{d}{dx} [2 \sec^2 x \tan x] = 2 (2 \sec^2 x \tan^2 x + \sec^4 x)$$

Denominator derivative:

$$\frac{d}{dx} (6x) = 6$$

Now take the limit:

$$\lim_{x \rightarrow 0} \frac{2(2 \sec^2 x \tan^2 x + \sec^4 x)}{6} = \frac{2(2 \cdot 1 \cdot 0 + 1)}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{\tan(3x)}$$

Question 10 - Solution

As $x \rightarrow 0$:

$$\sin(5x) \rightarrow 0, \quad \tan(3x) \rightarrow 0 \Rightarrow \text{Indeterminate form } \frac{0}{0}$$

Apply L'Hospital's Rule:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(5x)}{\tan(3x)} &= \lim_{x \rightarrow 0} \frac{d}{dx} \left[\frac{\sin(5x)}{\tan(3x)} \right] = \lim_{x \rightarrow 0} \frac{5 \cos(5x)}{3 \sec^2(3x)} \\ &= \lim_{x \rightarrow 0} \frac{5 \cos(5x)}{3 \cdot \frac{1}{\cos^2(3x)}} = \lim_{x \rightarrow 0} \frac{5 \cos(5x) \cos^2(3x)}{3} \end{aligned}$$

As $x \rightarrow 0$:

$$\cos(5x) \rightarrow 1, \quad \cos(3x) \rightarrow 1 \Rightarrow \cos^2(3x) \rightarrow 1$$

So,

$$\frac{5 \cdot 1 \cdot 1}{3} = \frac{5}{3}$$

Final Answer:

$$\boxed{\frac{5}{3}}$$