

Question 1

The position of an object moving along a straight line is given by:

$$s(t) = 3t^3 - 5t^2 + 2t \quad (\text{in meters, where } t \text{ is in seconds})$$

- (a) Find the velocity function $v(t)$.
- (b) Find the instantaneous velocity at $t = 2$ seconds.
- (c) Find the average velocity between $t = 1$ and $t = 2$.

Question 1 - Solution

We are given:

$$s(t) = 3t^3 - 5t^2 + 2t$$

(a) Velocity is the derivative of position:

$$v(t) = s'(t) = 9t^2 - 10t + 2$$

(b) Instantaneous velocity at $t = 2$:

$$v(2) = 9(4) - 10(2) + 2 = 36 - 20 + 2 = 18$$

Answer: $v(2) = 18 \text{ m/s}$

(c) Average velocity from $t = 1$ to $t = 2$:

$$\frac{s(2) - s(1)}{2 - 1} = s(2) - s(1)$$

Compute $s(2)$:

$$s(2) = 3(8) - 5(4) + 2(2) = 24 - 20 + 4 = 8$$

Compute $s(1)$:

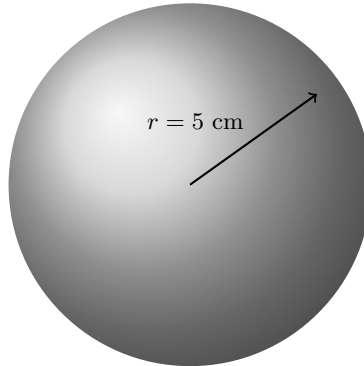
$$s(1) = 3(1) - 5(1) + 2(1) = 3 - 5 + 2 = 0$$

$$\text{Average velocity} = \frac{8 - 0}{1} = 8$$

Answer: 8 m/s

Question 2**Problem:**

A spherical balloon is being inflated so that its volume increases at a rate of $100 \text{ cm}^3/\text{sec}$. How fast is the radius increasing when the radius is 5 cm ?



Inflating sphere

Question 2 - Solution

We are given the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to time t :

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

We are told that:

$$\frac{dV}{dt} = 100, \quad r = 5$$

Substitute into the equation:

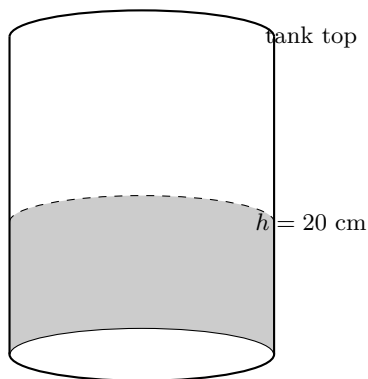
$$100 = 4\pi(5)^2 \cdot \frac{dr}{dt} \Rightarrow 100 = 100\pi \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi}$$

Answer:

$\frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec}$
--

Question 3**Problem:**

Water is being poured into a cylindrical tank at a rate of $500 \text{ cm}^3/\text{s}$. The tank has a radius of 10 cm. How fast is the water level rising when the water is 20 cm deep?



Question 3 - Solution

The volume of a cylinder is:

$$V = \pi r^2 h$$

Given: $\frac{dV}{dt} = 500$, $r = 10$, $h = 20$

We differentiate with respect to time:

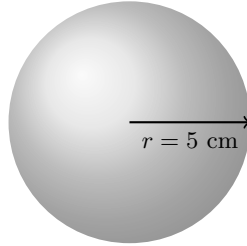
$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \Rightarrow 500 = \pi (10)^2 \frac{dh}{dt} \Rightarrow 500 = 100\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{500}{100\pi} = \frac{5}{\pi}$$

Answer:

$\frac{dh}{dt} = \frac{5}{\pi} \text{ cm/sec}$
--

Question 4**Problem:**

A spherical balloon is being inflated so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius increasing when the radius is 5 cm ?



Question 4 - Solution

We use the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to t :

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute:

$$\frac{dV}{dt} = 100, \quad r = 5$$

$$100 = 4\pi(5)^2 \frac{dr}{dt} \Rightarrow 100 = 100\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi}$$

Answer:

$\frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec}$
--

Question 5

Lighthouse Problem:

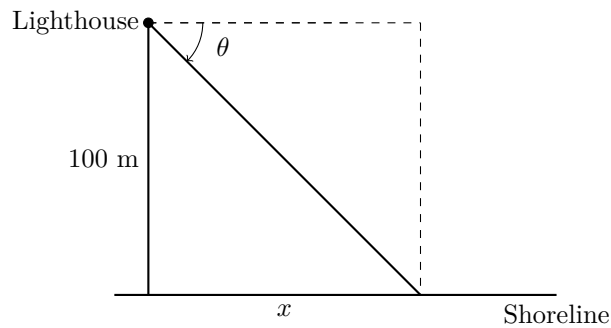
A lighthouse is located at the top of a cliff that is 100 meters high. The shoreline is straight and lies in a vertical plane. The beam of light rotates in this plane at a constant rate of

$$\frac{d\theta}{dt} = 0.05 \text{ rad/s.}$$

Let x be the distance along the shoreline from the base of the cliff to the point where the light hits the shore.

How fast is the spot of light moving along the shoreline when

$$\theta = \frac{\pi}{4} ?$$



Question 5 - Solution

Let:

- θ be the angle between the light beam and the horizontal shoreline.
- x be the distance along the shoreline to the spot of light.

From the right triangle formed by the beam, the cliff, and the shoreline:

$$\tan(\theta) = \frac{x}{100} \quad \Rightarrow \quad x = 100 \tan(\theta)$$

Differentiate both sides with respect to time t :

$$\frac{dx}{dt} = 100 \sec^2(\theta) \frac{d\theta}{dt}$$

Substitute the given values:

$$\theta = \frac{\pi}{4}, \quad \frac{d\theta}{dt} = 0.05 \text{ rad/s}, \quad \sec^2\left(\frac{\pi}{4}\right) = 2$$

$$\frac{dx}{dt} = 100(2)(0.05) = 10 \text{ m/s}$$

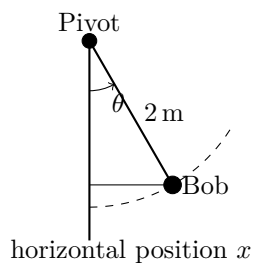
Answer:

$\frac{dx}{dt} = 10 \text{ m/s}$

Question 6

Problem:

A pendulum of length 2 meters swings back and forth in a vertical plane. At a certain instant, the angle θ between the pendulum and the vertical is $\frac{\pi}{6}$, and the angle is increasing at a rate of 0.4 rad/s . How fast is the pendulum bob moving horizontally at that instant?



Question 6 - Solution

Let:

- θ be the angle between the pendulum and vertical
- x be the horizontal displacement of the bob from the vertical

Using trigonometry:

$$x = L \sin(\theta) = 2 \sin(\theta)$$

Differentiate both sides with respect to time t :

$$\frac{dx}{dt} = 2 \cos(\theta) \cdot \frac{d\theta}{dt}$$

Substitute:

$$\theta = \frac{\pi}{6}, \quad \frac{d\theta}{dt} = 0.4 \text{ rad/s}, \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{dx}{dt} = 2 \cdot \frac{\sqrt{3}}{2} \cdot 0.4 = \sqrt{3} \cdot 0.4 \approx 0.6928 \text{ m/s}$$

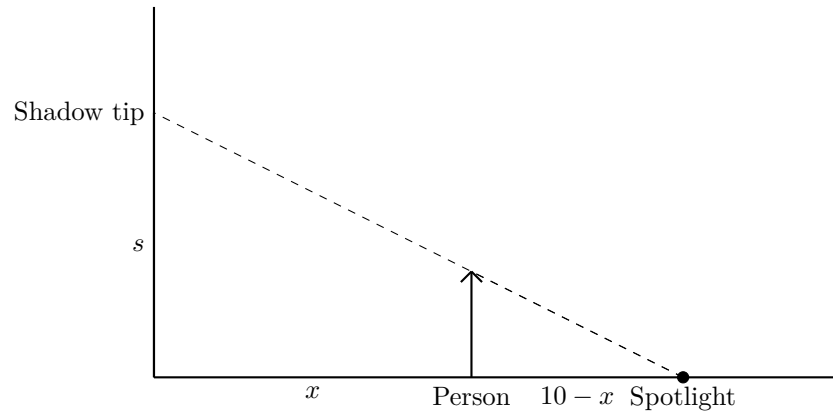
Answer:

$\frac{dx}{dt} \approx 0.693 \text{ m/s}$

Question 7

Problem:

A spotlight is located on the ground 10 meters from a vertical wall. A person 2 meters tall walks away from the wall at a speed of 1.5 m/s. How fast is the length of their shadow on the wall increasing when the person is 6 meters from the wall?



Question 7 - Solution

Let:

- x : distance from person to wall (in m)
- s : length of the shadow on the wall (in m)
- Person's height: 2 m
- Spotlight is 10 m from the wall

Using similar triangles (person is between wall and spotlight, so spotlight-person distance is $10 - x$):

$$\frac{2}{10 - x} = \frac{s}{10} \quad \Rightarrow \quad s = \frac{20}{10 - x}$$

Differentiate with respect to t :

$$\frac{ds}{dt} = 20 \cdot \frac{d}{dt} ((10 - x)^{-1}) = 20(10 - x)^{-2} \cdot \frac{dx}{dt}$$

Substitute $x = 6$ and $\frac{dx}{dt} = 1.5$:

$$\frac{ds}{dt} = \frac{20(1.5)}{(10 - 6)^2} = \frac{30}{16} = \frac{15}{8}$$

Answer:

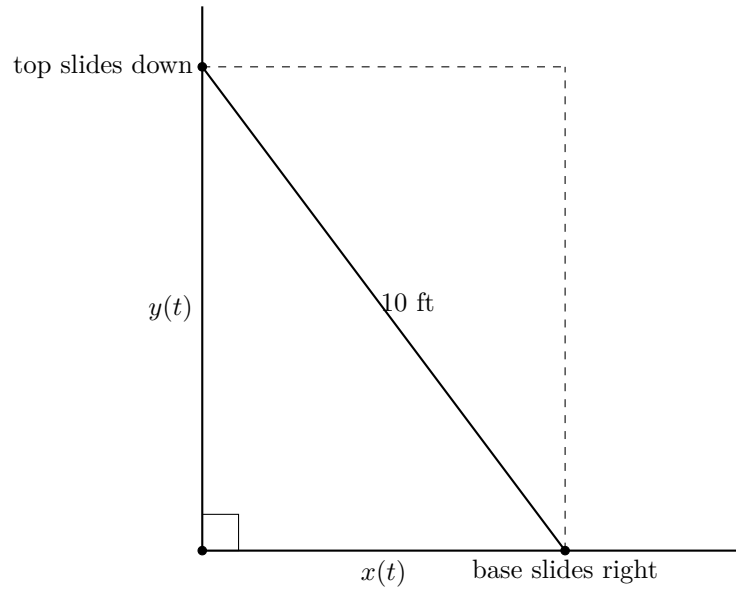
$\frac{ds}{dt} = \frac{15}{8} \text{ m/s}$
--

 (The shadow is increasing at this rate.)

Question 8

Problem:

A 10-foot ladder is leaning against a vertical wall. The bottom of the ladder is sliding away from the wall at a rate of 2 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?



Question 8 - Solution

Let:

- $x(t)$: distance from the wall to the base of the ladder
- $y(t)$: height of the top of the ladder on the wall
- Given: $\frac{dx}{dt} = 2 \text{ ft/s}$, $x = 6$

Using the Pythagorean theorem:

$$x^2 + y^2 = 10^2 = 100$$

Differentiate both sides with respect to time t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

We know:

$$x = 6, \quad \frac{dx}{dt} = 2$$

Solve for y :

$$36 + y^2 = 100 \Rightarrow y^2 = 64 \Rightarrow y = 8$$

Substitute into the differentiated equation:

$$6(2) + 8 \frac{dy}{dt} = 0 \Rightarrow 12 + 8 \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{12}{8} = -\frac{3}{2}$$

Answer:

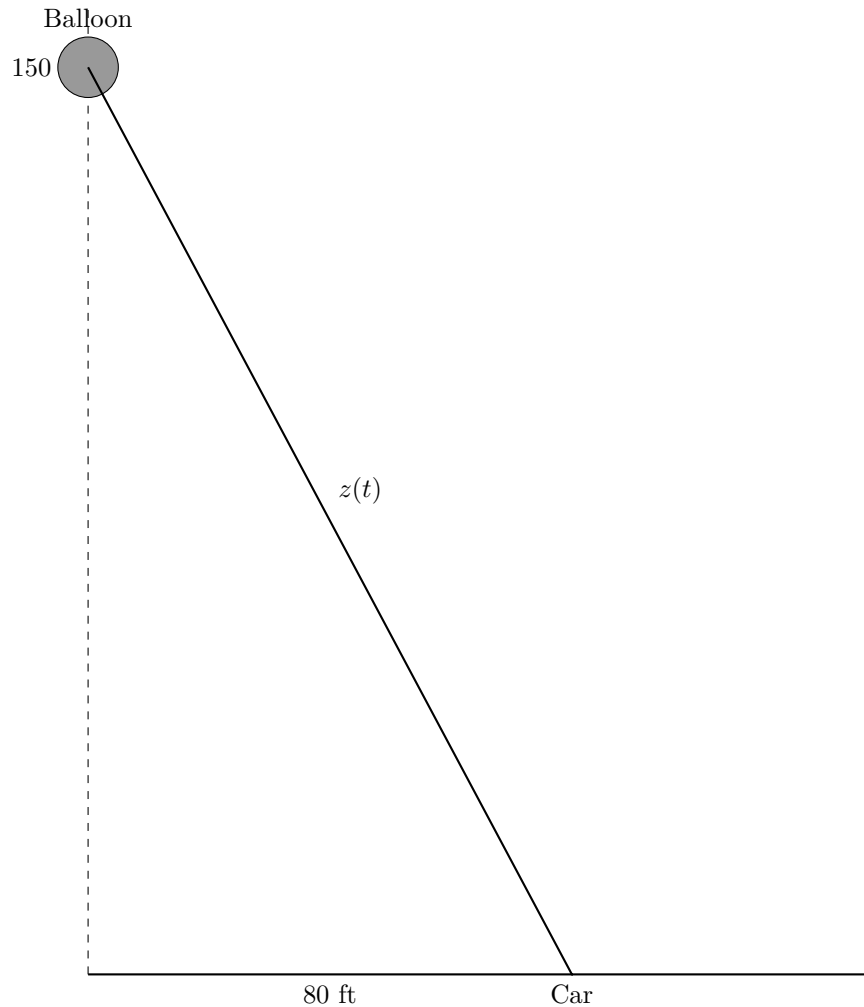
$$\boxed{\frac{dy}{dt} = -\frac{3}{2} \text{ ft/s}}$$

The negative sign indicates that the top of the ladder is sliding downward.

Question 9

Problem:

A balloon is rising vertically at a constant rate of 5 ft/s, and at the same time, a car drives directly away from the point on the ground beneath the balloon at 60 ft/s. How fast is the distance between the car and the balloon increasing when the car is 80 feet from the balloon's launch point and the balloon is 150 feet high?



Question 9 - Solution

Let:

- $x(t)$: distance from car to balloon launch point (horizontal) — $\frac{dx}{dt} = 60$
- $y(t)$: height of balloon — $\frac{dy}{dt} = 5$
- $z(t)$: distance between car and balloon (hypotenuse)

We apply the Pythagorean theorem:

$$z^2 = x^2 + y^2$$

Differentiate both sides:

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

Plug in values:

$$x = 80, \quad y = 150 \Rightarrow z = \sqrt{80^2 + 150^2} = \sqrt{6400 + 22500} = \sqrt{28900} = 170$$

Now plug in:

$$170 \cdot \frac{dz}{dt} = 80 \cdot 60 + 150 \cdot 5 \Rightarrow 170 \cdot \frac{dz}{dt} = 4800 + 750 = 5550 \Rightarrow \frac{dz}{dt} = \frac{5550}{170} = 32.65 \text{ ft/s}$$

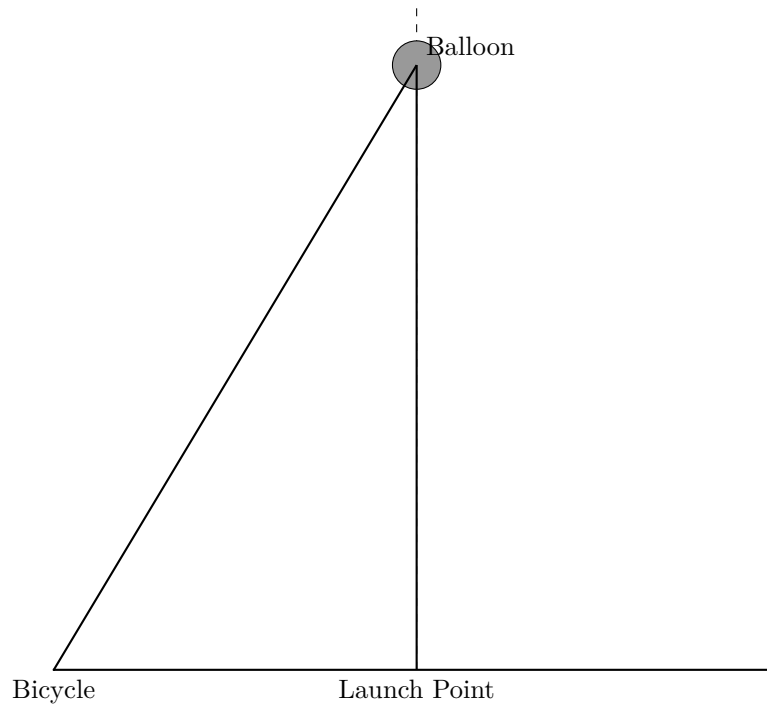
Answer:

$\frac{dz}{dt} = 32.65 \text{ ft/s}$

Question 10

Problem:

A hot air balloon is descending vertically at a rate of 2 ft/s, while a bicyclist passes directly beneath it, traveling in a straight line at a speed of 15 ft/s. How fast is the distance between the bicyclist and the balloon changing when the bicyclist is 60 feet from the point directly below the balloon and the balloon is 100 feet above the ground?



Question 10 - Solution

Let:

- $x(t)$: horizontal distance of bicycle from point below the balloon — $\frac{dx}{dt} = 15$
- $y(t)$: vertical distance of balloon from ground — $\frac{dy}{dt} = -2$
- $z(t)$: distance between balloon and bicycle

Using the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiate both sides with respect to time t :

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

Plug in known values:

$$x = 60, \quad y = 100, \quad z = \sqrt{60^2 + 100^2} = \sqrt{3600 + 10000} = \sqrt{13600} = 116.6$$

$$z \frac{dz}{dt} = 60(15) + 100(-2) = 900 - 200 = 700 \Rightarrow \frac{dz}{dt} = \frac{700}{116.6} \approx 6.01 \text{ ft/s}$$

Answer:

$\frac{dz}{dt} \approx 6.01 \text{ ft/s}$