

Question 1

Differentiate the function:

$$f(x) = \sqrt{1 + e^{3x^2}}$$

Clearly show how the chain rule is used in multiple layers.

Question 1 - Solution

We are given:

$$f(x) = \sqrt{1 + e^{3x^2}} = (1 + e^{3x^2})^{1/2}$$

Use the chain rule:

$$\begin{aligned} f'(x) &= \frac{1}{2}(1 + e^{3x^2})^{-1/2} \cdot \frac{d}{dx}(1 + e^{3x^2}) \\ &= \frac{1}{2\sqrt{1 + e^{3x^2}}} \cdot \frac{d}{dx}(e^{3x^2}) \end{aligned}$$

Now apply the chain rule again to e^{3x^2} :

$$\frac{d}{dx}(e^{3x^2}) = e^{3x^2} \cdot \frac{d}{dx}(3x^2) = e^{3x^2} \cdot 6x$$

Putting it all together:

$$f'(x) = \frac{1}{2\sqrt{1 + e^{3x^2}}} \cdot e^{3x^2} \cdot 6x = \boxed{\frac{3x \cdot e^{3x^2}}{\sqrt{1 + e^{3x^2}}}}$$

Question 2

Let $f(x) = \sin(\sqrt{1+x^4})$.

- (a) Find $f'(x)$ using the chain rule.
- (b) Evaluate $f'(1)$ exactly (do not approximate numerically).

Question 2 - Solution

We are given:

$$f(x) = \sin\left(\sqrt{1+x^4}\right)$$

Let's apply the chain rule step by step.

Step 1: Let

$$u(x) = \sqrt{1+x^4} = (1+x^4)^{1/2} \quad \text{so that} \quad f(x) = \sin(u(x))$$

Step 2: Apply the chain rule:

$$f'(x) = \cos(u(x)) \cdot u'(x)$$

Step 3: Differentiate $u(x)$:

$$u(x) = (1+x^4)^{1/2} \Rightarrow u'(x) = \frac{1}{2}(1+x^4)^{-1/2} \cdot 4x^3 = \frac{2x^3}{\sqrt{1+x^4}}$$

Final derivative:

$$f'(x) = \cos\left(\sqrt{1+x^4}\right) \cdot \frac{2x^3}{\sqrt{1+x^4}}$$

$$\boxed{f'(x) = \frac{2x^3 \cos\left(\sqrt{1+x^4}\right)}{\sqrt{1+x^4}}}$$

(b) Evaluate at $x = 1$:

$$f'(1) = \frac{2(1)^3 \cdot \cos\left(\sqrt{1+1^4}\right)}{\sqrt{1+1^4}} = \frac{2 \cdot \cos(\sqrt{2})}{\sqrt{2}}$$

$$\boxed{f'(1) = \frac{2 \cos(\sqrt{2})}{\sqrt{2}}}$$

Question 3

Let $y = \ln(\tan^3(5x^2 + 1))$.

- (a) Differentiate y with respect to x using the chain rule.
- (b) Simplify the derivative as much as possible.

Question 3 - Solution

We are given:

$$y = \ln(\tan^3(5x^2 + 1))$$

Step 1: Use logarithmic identity:

$$\ln(\tan^3(5x^2 + 1)) = 3 \ln(\tan(5x^2 + 1))$$

Step 2: Differentiate using chain rule:

$$\frac{dy}{dx} = 3 \cdot \frac{1}{\tan(5x^2 + 1)} \cdot \sec^2(5x^2 + 1) \cdot (10x)$$

Step 3: Simplify the expression:

$$\frac{dy}{dx} = \frac{30x \sec^2(5x^2 + 1)}{\tan(5x^2 + 1)}$$

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Question 4

Let $y = \sqrt{1 + \sin^2(3x)}$.

- (a) Find $\frac{dy}{dx}$ using the chain rule.
- (b) Simplify the result as much as possible.

Question 4 - Solution

We are given:

$$y = \sqrt{1 + \sin^2(3x)} = (1 + \sin^2(3x))^{1/2}$$

Step 1: Apply the chain rule:

Let: - Outer function: $u^{1/2}$ - Inner function: $u = 1 + \sin^2(3x)$

We compute:

$$\frac{dy}{dx} = \frac{1}{2}(1 + \sin^2(3x))^{-1/2} \cdot \frac{d}{dx}[\sin^2(3x)]$$

Now differentiate $\sin^2(3x)$ using the chain rule:

$$\frac{d}{dx}[\sin^2(3x)] = 2 \sin(3x) \cdot \cos(3x) \cdot 3 = 6 \sin(3x) \cos(3x)$$

Step 2: Combine terms:

$$\frac{dy}{dx} = \frac{1}{2}(1 + \sin^2(3x))^{-1/2} \cdot 6 \sin(3x) \cos(3x)$$

$$\frac{dy}{dx} = \frac{3 \sin(3x) \cos(3x)}{\sqrt{1 + \sin^2(3x)}}$$

$$\boxed{\frac{dy}{dx} = \frac{3 \sin(3x) \cos(3x)}{\sqrt{1 + \sin^2(3x)}}}$$

Question 5

Let $f(x) = e^{\tan(x^3+1)}$.

- (a) Use the chain rule to find $f'(x)$.
- (b) Clearly identify each function used in the composition and the order in which the chain rule is applied.

Question 5 - Solution

We are given:

$$f(x) = e^{\tan(x^3+1)}$$

This is a composition of multiple functions. Let's break it down:

- Outer function: $u = e^v$
- Middle function: $v = \tan(w)$
- Inner function: $w = x^3 + 1$

Using the chain rule:

$$f'(x) = \frac{d}{dx} \left[e^{\tan(x^3+1)} \right] = e^{\tan(x^3+1)} \cdot \frac{d}{dx} [\tan(x^3 + 1)]$$

Now,

$$\frac{d}{dx} [\tan(x^3 + 1)] = \sec^2(x^3 + 1) \cdot \frac{d}{dx} [x^3 + 1] = \sec^2(x^3 + 1) \cdot 3x^2$$

Putting it all together:

$$f'(x) = e^{\tan(x^3+1)} \cdot \sec^2(x^3 + 1) \cdot 3x^2$$

$$\boxed{f'(x) = 3x^2 \cdot \sec^2(x^3 + 1) \cdot e^{\tan(x^3+1)}}$$

Question 6

Let $f(x) = \ln\left(\sqrt{1 + \sin^2(5x)}\right)$.

- (a) Use the chain rule to compute $f'(x)$.
- (b) Identify the layers of composition involved and describe the differentiation process step-by-step.

Question 6 - Solution

We are given:

$$f(x) = \ln \left(\sqrt{1 + \sin^2(5x)} \right)$$

Let's simplify before differentiating:

$$f(x) = \ln \left((1 + \sin^2(5x))^{1/2} \right) = \frac{1}{2} \ln(1 + \sin^2(5x))$$

Now we differentiate:

$$f'(x) = \frac{1}{2} \cdot \frac{1}{1 + \sin^2(5x)} \cdot \frac{d}{dx}[\sin^2(5x)]$$

We apply the chain rule to $\sin^2(5x)$:

$$\frac{d}{dx}[\sin^2(5x)] = 2 \sin(5x) \cdot \cos(5x) \cdot 5 = 10 \sin(5x) \cos(5x)$$

Therefore:

$$f'(x) = \frac{1}{2} \cdot \frac{10 \sin(5x) \cos(5x)}{1 + \sin^2(5x)}$$

$$f'(x) = \frac{5 \sin(5x) \cos(5x)}{1 + \sin^2(5x)}$$

Question 7

Let $y = e^{\tan^3(2x)}$.

- (a) Use the chain rule to compute $\frac{dy}{dx}$.
- (b) Clearly outline the layers of composition and how each derivative is applied.

Question 7 - Solution

We are given:

$$y = e^{\tan^3(2x)} = e^{(\tan(2x))^3}$$

Let's identify the layers of composition:

- Outermost: e^u
- Middle: $u = v^3$
- Inner: $v = \tan(2x)$
- Innermost: $2x$

Apply the chain rule step-by-step:

$$\frac{dy}{dx} = \frac{d}{dx} \left(e^{(\tan(2x))^3} \right) = e^{(\tan(2x))^3} \cdot \frac{d}{dx} ((\tan(2x))^3)$$

Differentiate the power:

$$\frac{d}{dx} ((\tan(2x))^3) = 3(\tan(2x))^2 \cdot \frac{d}{dx} [\tan(2x)]$$

Differentiate $\tan(2x)$:

$$\frac{d}{dx} [\tan(2x)] = \sec^2(2x) \cdot 2$$

Now put it all together:

$$\frac{dy}{dx} = e^{(\tan(2x))^3} \cdot 3(\tan(2x))^2 \cdot \sec^2(2x) \cdot 2$$

Simplify:

$$\boxed{\frac{dy}{dx} = 6(\tan(2x))^2 \sec^2(2x) \cdot e^{(\tan(2x))^3}}$$

Question 8

Let $y = \ln\left(\sqrt{1 + e^{3x^2}}\right)$.

- (a) Use the chain rule to compute $\frac{dy}{dx}$.
- (b) Explain how the composition of functions influences the order of derivatives.

Question 8 - Solution

We are given:

$$y = \ln \left(\sqrt{1 + e^{3x^2}} \right)$$

We rewrite the square root as a power:

$$y = \ln \left((1 + e^{3x^2})^{1/2} \right)$$

Using the log rule:

$$y = \frac{1}{2} \ln(1 + e^{3x^2})$$

Differentiate:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{3x^2}} \cdot \frac{d}{dx}[e^{3x^2}]$$

Differentiate the exponential:

$$\frac{d}{dx}[e^{3x^2}] = e^{3x^2} \cdot \frac{d}{dx}[3x^2] = e^{3x^2} \cdot 6x$$

Putting it all together:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{3x^2}} \cdot e^{3x^2} \cdot 6x$$

Simplify:

$$\boxed{\frac{dy}{dx} = \frac{3xe^{3x^2}}{1 + e^{3x^2}}}$$

Explanation: This function has a nested composition: Innermost: $3x^2$, Exponential: e^{3x^2} , Added to 1, Square root $\rightarrow$ converted to a power, Then natural logarithm

At each step, the derivative is taken from the outside and multiplied by the derivative of the inside, following the chain rule.

Question 9

Let $f(x) = \sin^2(5x^3 + 1)$.

- (a) Use the chain rule to compute $f'(x)$.
- (b) Write down each layer of function composition clearly before differentiating.

Question 9 - Solution

We are given:

$$f(x) = \sin^2(5x^3 + 1)$$

This can be rewritten as:

$$f(x) = [\sin(5x^3 + 1)]^2$$

Layers of function composition:

1. Innermost function: $u(x) = 5x^3 + 1$ 2. Middle function: $v(u) = \sin(u)$ 3. Outer function: $f(v) = v^2$

Differentiate step by step:

Start from the outside:

$$f'(x) = 2 \cdot \sin(5x^3 + 1) \cdot \frac{d}{dx}[\sin(5x^3 + 1)]$$

Now differentiate the inner sine function:

$$\frac{d}{dx}[\sin(5x^3 + 1)] = \cos(5x^3 + 1) \cdot \frac{d}{dx}[5x^3 + 1] = \cos(5x^3 + 1) \cdot 15x^2$$

Putting it all together:

$$f'(x) = 2 \cdot \sin(5x^3 + 1) \cdot \cos(5x^3 + 1) \cdot 15x^2$$

$$\boxed{f'(x) = 30x^2 \cdot \sin(5x^3 + 1) \cdot \cos(5x^3 + 1)}$$

Alternatively, you may express the result using the identity:

$$2 \sin \theta \cos \theta = \sin(2\theta)$$

$$\Rightarrow f'(x) = 15x^2 \cdot \sin(2(5x^3 + 1)) = \boxed{15x^2 \cdot \sin(10x^3 + 2)}$$

Either boxed expression is valid.

Question 10

Let $y = \ln(\sqrt{1 + e^{4x}})$.

- (a) Use the chain rule to find $\frac{dy}{dx}$.
- (b) Clearly indicate the nested functions and show each step of differentiation.

Question 10 - Solution

We are given:

$$y = \ln \left(\sqrt{1 + e^{4x}} \right)$$

First, simplify the expression:

$$y = \ln \left[(1 + e^{4x})^{1/2} \right] = \frac{1}{2} \ln(1 + e^{4x})$$

Now differentiate:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1 + e^{4x}} \cdot \frac{d}{dx}(1 + e^{4x})$$

Differentiate the inner exponential:

$$\frac{d}{dx}(1 + e^{4x}) = 4e^{4x}$$

Putting it all together:

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{4e^{4x}}{1 + e^{4x}} = \boxed{\frac{2e^{4x}}{1 + e^{4x}}}$$

Nested Function Structure:

$$u = e^{4x}, v = 1 + u, w = \ln(\sqrt{v}) = \frac{1}{2} \ln(v)$$

Differentiation followed this path:

$$\frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$