

Question 1

Differentiate the function:

$$f(x) = \sinh(x) \cdot \cosh(x)$$

Then express your answer in terms of a single hyperbolic function if possible.

Question 1 - Solution

We are given:

$$f(x) = \sinh(x) \cdot \cosh(x)$$

Use the product rule:

$$f'(x) = \cosh(x) \cdot \cosh(x) + \sinh(x) \cdot \sinh(x) = \cosh^2(x) + \sinh^2(x)$$

Recall the identity:

$$\cosh^2(x) - \sinh^2(x) = 1 \quad \Rightarrow \quad \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1$$

Thus:

$$f'(x) = \boxed{\cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1}$$

Answer: $f'(x) = \boxed{\cosh(2x)}$

Question 2

Let $f(x) = \frac{\sinh(3x)}{\cosh(x)}$.

- (a) Compute $f'(x)$ using differentiation rules.
- (b) Simplify the result as much as possible using hyperbolic identities.

Question 2 - Solution

The quotient rule gives

$$f'(x) = \frac{3 \cosh(3x) \cosh x - \sinh(3x) \sinh x}{\cosh^2 x}.$$

Use the product identities

$$\cosh(3x) \cosh x = \frac{1}{2}(\cosh 4x + \cosh 2x),$$

$$\sinh(3x) \sinh x = \frac{1}{2}(\cosh 4x - \cosh 2x).$$

Thus

$$f'(x) = \frac{\cosh 4x + 2 \cosh 2x}{\cosh^2 x}.$$

At zero the derivative is 3, agreeing with the original quotient-rule expression.

Question 3

Let $f(x) = \tanh(x) \sinh(x)$.

- (a) Compute $f'(x)$.
- (b) Simplify your result as much as possible using hyperbolic identities.

Question 3 - Solution

We are given:

$$f(x) = \tanh(x) \sinh(x)$$

This is a product of two functions, so we apply the product rule:

$$f'(x) = \tanh'(x) \sinh(x) + \tanh(x) \sinh'(x)$$

Recall the derivatives:

$$\tanh'(x) = \operatorname{sech}^2(x), \quad \sinh'(x) = \cosh(x)$$

Substitute:

$$f'(x) = \operatorname{sech}^2(x) \sinh(x) + \tanh(x) \cosh(x)$$

Now simplify. Recall:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \Rightarrow \tanh(x) \cosh(x) = \sinh(x)$$

So:

$$f'(x) = \operatorname{sech}^2(x) \sinh(x) + \sinh(x) = \sinh(x) (\operatorname{sech}^2(x) + 1)$$

Final Answer:

$$\boxed{f'(x) = \sinh(x) (\operatorname{sech}^2(x) + 1)}$$

Question 4

Let $f(x) = \frac{\cosh^2(x)-1}{\sinh(x)}$.

- (a) Simplify the expression before differentiating.
- (b) Compute $f'(x)$.
- (c) Identify any points where $f'(x)$ is undefined.

Question 4 - Solution**(a) Simplification:**

We start with:

$$f(x) = \frac{\cosh^2(x) - 1}{\sinh(x)}$$

Use the identity:

$$\cosh^2(x) - 1 = \sinh^2(x)$$

Thus:

$$f(x) = \frac{\sinh^2(x)}{\sinh(x)} = \sinh(x)$$

(b) Differentiate:

$$f'(x) = \frac{d}{dx} \sinh(x) = \cosh(x)$$

(c) Points of Non-Differentiability:

Since $f(x) = \sinh(x)$ is differentiable for all real x , and the simplification is valid as long as the denominator of the original function $\sinh(x) \neq 0$, we must consider the domain of the original function.

The original function is undefined at:

$$\sinh(x) = 0 \Rightarrow x = 0$$

So although $f(x) = \sinh(x)$ is differentiable everywhere, the original form $f(x) = \frac{\cosh^2(x)-1}{\sinh(x)}$ is undefined at $x = 0$, and thus f is not defined or differentiable there in its original form.

Conclusion:

$f'(x) = \cosh(x), \quad f'(x) \text{ undefined at } x = 0$

Question 5

Let $f(x) = \frac{\sinh(x)}{\cosh(x)}$.

- (a) Simplify the function if possible.
- (b) Find $f'(x)$.
- (c) Determine the intervals where $f(x)$ is increasing or decreasing.

Question 5 - Solution**(a) Simplify the Function:**

We have:

$$f(x) = \frac{\sinh(x)}{\cosh(x)}$$

This is the definition of the hyperbolic tangent function:

$$f(x) = \tanh(x)$$

(b) Differentiate $f(x) = \tanh(x)$:

The derivative of $\tanh(x)$ is:

$$f'(x) = \operatorname{sech}^2(x)$$

(c) Increasing/Decreasing Intervals:

Since $f'(x) = \operatorname{sech}^2(x) > 0$ for all real x , the function is strictly increasing on its entire domain.

Conclusion:

$f'(x) = \operatorname{sech}^2(x), \quad f(x) \text{ is increasing on } (-\infty, \infty)$
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Question 6

Let $f(x) = x \cdot \cosh(x^2)$.

- (a) Find $f'(x)$.
- (b) Determine the value of the derivative at $x = 0$.
- (c) Is there a local minimum, maximum, or neither at $x = 0$? Justify using the derivative.

Question 6 - Solution

We are given:

$$f(x) = x \cdot \cosh(x^2)$$

(a) Find $f'(x)$:

This is a product of two functions: Let $u = x$, $v = \cosh(x^2)$

Then:

$$f'(x) = u'v + uv' = 1 \cdot \cosh(x^2) + x \cdot \frac{d}{dx}[\cosh(x^2)]$$

Differentiate $\cosh(x^2)$ using the chain rule:

$$\frac{d}{dx}[\cosh(x^2)] = \sinh(x^2) \cdot 2x$$

So:

$$f'(x) = \cosh(x^2) + x \cdot (2x \cdot \sinh(x^2)) = \cosh(x^2) + 2x^2 \sinh(x^2)$$

$$\boxed{f'(x) = \cosh(x^2) + 2x^2 \sinh(x^2)}$$

(b) Value at $x = 0$:

$$f'(0) = \cosh(0^2) + 2 \cdot 0^2 \cdot \sinh(0^2) = \cosh(0) = 1$$

$$\boxed{f'(0) = 1}$$

(c) Local Behavior at $x = 0$:

Since $f'(0) = 1 > 0$, the function is increasing at $x = 0$, so:

Conclusion: There is **no local minimum or maximum** at $x = 0$, because the derivative is positive and the function is increasing there.

Question 7

Let $f(x) = \tanh(x^2) \cdot \operatorname{sech}(x)$.

- (a) Differentiate $f(x)$ using the product rule and the chain rule.
- (b) Simplify the derivative as much as possible using hyperbolic identities.

Question 7 - Solution

We are given:

$$f(x) = \tanh(x^2) \cdot \operatorname{sech}(x)$$

(a) Use the product rule:

Let:

$$u = \tanh(x^2), \quad v = \operatorname{sech}(x)$$

Then:

$$f'(x) = u'v + uv'$$

Differentiate each part:

Step 1: $u = \tanh(x^2) \Rightarrow u' = \operatorname{sech}^2(x^2) \cdot 2x$ using chain rule

Step 2: $v = \operatorname{sech}(x) \Rightarrow v' = -\operatorname{sech}(x) \cdot \tanh(x)$

Now substitute into the product rule:

$$f'(x) = [\operatorname{sech}^2(x^2) \cdot 2x] \cdot \operatorname{sech}(x) + \tanh(x^2) \cdot [-\operatorname{sech}(x) \cdot \tanh(x)]$$

(b) Final expression:

$$f'(x) = 2x \operatorname{sech}^2(x^2) \operatorname{sech}(x) - \tanh(x^2) \operatorname{sech}(x) \tanh(x)$$

Question 8

Let $f(x) = \frac{\sinh(3x)}{x^2+1}$.

- (a) Find the derivative $f'(x)$ using the quotient rule.
- (b) Evaluate the derivative at $x = 0$.

Question 8 - Solution

We are given:

$$f(x) = \frac{\sinh(3x)}{x^2 + 1}$$

(a) Find $f'(x)$:

This is a quotient, so we use the quotient rule:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

Let:

$$g(x) = \sinh(3x), \quad h(x) = x^2 + 1$$

Then:

$$g'(x) = 3 \cosh(3x), \quad h'(x) = 2x$$

So:

$$f'(x) = \frac{3 \cosh(3x)(x^2 + 1) - \sinh(3x)(2x)}{(x^2 + 1)^2}$$

$$f'(x) = \frac{3(x^2 + 1) \cosh(3x) - 2x \sinh(3x)}{(x^2 + 1)^2}$$

(b) Evaluate at $x = 0$:

First note:

$$\cosh(0) = 1, \quad \sinh(0) = 0$$

Then:

$$f'(0) = \frac{3(0^2 + 1) \cdot 1 - 0}{(0^2 + 1)^2} = \frac{3}{1} = 3$$

$$f'(0) = 3$$

Question 9

Let $f(x) = \tanh(x^2 + 1)$.

- (a) Find the derivative $f'(x)$ using the chain rule and hyperbolic identities.
- (b) Determine the value of the derivative at $x = 0$.

Question 9 - Solution

We are given:

$$f(x) = \tanh(x^2 + 1)$$

(a) Find $f'(x)$:

Use the chain rule:

$$f'(x) = \frac{d}{dx} [\tanh(u)] \cdot \frac{d}{dx} (x^2 + 1)$$

Recall:

$$\frac{d}{dx} [\tanh(u)] = \operatorname{sech}^2(u)$$

So let $u = x^2 + 1$, then:

$$f'(x) = \operatorname{sech}^2(x^2 + 1) \cdot 2x$$

$$\boxed{f'(x) = 2x \cdot \operatorname{sech}^2(x^2 + 1)}$$

(b) Evaluate at $x = 0$:

$$f'(0) = 2 \cdot 0 \cdot \operatorname{sech}^2(0^2 + 1) = 0$$

$$\boxed{f'(0) = 0}$$

Question 10

Let $f(x) = \frac{\sinh(3x)}{\cosh(x)}$.

- (a) Differentiate $f(x)$ using the quotient rule and known derivatives of hyperbolic functions.
- (b) Simplify the expression for $f'(x)$ as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \frac{\sinh(3x)}{\cosh(x)}$$

(a) Apply the quotient rule:

The quotient rule is:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Let:

$$u = \sinh(3x), \quad v = \cosh(x)$$

Differentiate:

$$u' = 3 \cosh(3x), \quad v' = \sinh(x)$$

Apply the rule:

$$f'(x) = \frac{3 \cosh(3x) \cosh(x) - \sinh(3x) \sinh(x)}{\cosh^2(x)}$$

(b) Final simplified expression:

$f'(x) = \frac{3 \cosh(3x) \cosh(x) - \sinh(3x) \sinh(x)}{\cosh^2(x)}$
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