

Question 1

Differentiate the function:

$$f(x) = x \cdot \arcsin(x)$$

Then evaluate $f'(x)$ at $x = \frac{1}{2}$.

Question 1 - Solution

We are given:

$$f(x) = x \cdot \arcsin(x)$$

Use the product rule:

$$f'(x) = \arcsin(x) + x \cdot \frac{1}{\sqrt{1-x^2}}$$

Answer:

$$f'(x) = \boxed{\arcsin(x) + \frac{x}{\sqrt{1-x^2}}}$$

Now evaluate at $x = \frac{1}{2}$:

$$f'\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) + \frac{\frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\pi}{6} + \frac{1/2}{\sqrt{3}/2} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}$$

Answer: $f'\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6} + \frac{1}{\sqrt{3}}}$

Question 2

Let

$$f(x) = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

- (a) Show that $f(x) = 2 \tan^{-1}(x)$ for all $x \in (-1, 1)$
- (b) Use the identity in part (a) to find $f'(x)$.

Question 2 - Solution

(a) Use an identity for tangent of double angle:

We recall:

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Let $\theta = \tan^{-1}(x) \Rightarrow \tan(\theta) = x$. Then:

$$\tan(2\theta) = \frac{2x}{1 - x^2} \Rightarrow 2\theta = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

So:

$$f(x) = \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = 2 \tan^{-1}(x) \quad \text{for } x \in (-1, 1)$$

(b) Differentiate using identity:

From (a), $f(x) = 2 \tan^{-1}(x)$

Use the derivative:

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2} \Rightarrow f'(x) = 2 \cdot \frac{1}{1 + x^2} = \boxed{\frac{2}{1 + x^2}}$$

Question 3

Let

$$f(x) = \arcsin\left(\frac{2x}{1+x^2}\right), \quad x > 0.$$

- (a) Express f in terms of $\arctan x$, separately for $0 < x \leq 1$ and $x > 1$.
- (b) Find $f'(x)$ wherever it exists, and analyze $x = 1$.

Question 3 - Solution

Put $t = \arctan x$. For $x > 0$, $0 < t < \pi/2$ and

$$\frac{2x}{1+x^2} = \sin(2t).$$

The principal range of $\arcsin$ is $[-\pi/2, \pi/2]$. Therefore

$$f(x) = \begin{cases} 2 \arctan x, & 0 < x \leq 1, \\ \pi - 2 \arctan x, & x > 1. \end{cases}$$

Differentiating on the two open intervals gives

$$f'(x) = \begin{cases} 2/(1+x^2), & 0 < x < 1, \\ -2/(1+x^2), & x > 1. \end{cases}$$

At $x = 1$ the left and right derivatives are 1 and -1 , respectively, so no derivative exists there.

Question 4

Let

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

- (a) Determine the domain of $f(x)$.
- (b) Compute $f'(x)$ for $x \in (-1, 1)$.

Question 4 - Solution**(a) Domain:**

Since $f(x) = \tan^{-1}(\sqrt{1-x^2})$, we require the expression under the square root to be non-negative:

$$1 - x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow -1 \leq x \leq 1$$

But the square root function $\sqrt{1-x^2}$ is only real for $x \in [-1, 1]$, and since it appears inside an inverse tangent (which is defined for all real inputs), the domain of f is:

$$\boxed{[-1, 1]}$$

(b) Compute the derivative:

Let's differentiate using the chain rule:

$$f(x) = \tan^{-1}(u), \quad \text{where } u = \sqrt{1-x^2} \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

Now compute $\frac{du}{dx}$ where $u = \sqrt{1-x^2}$:

$$\frac{du}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Next, compute $1+u^2 = 1+(1-x^2) = 2-x^2$

So:

$$f'(x) = \frac{1}{2-x^2} \cdot \left(\frac{-x}{\sqrt{1-x^2}} \right) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

Question 5

Let

$$f(x) = \sec^{-1} \left(\frac{2x+3}{x-1} \right)$$

- (a) Determine the domain of $f(x)$.
- (b) Compute $f'(x)$.

Question 5 - Solution

(a) Domain:

The inverse secant function is defined for $|u| \geq 1$. Thus, we require

$$\left| \frac{2x+3}{x-1} \right| \geq 1$$

Squaring both sides:

$$\frac{(2x+3)^2}{(x-1)^2} \geq 1 \quad \Rightarrow \quad (2x+3)^2 \geq (x-1)^2$$

Expanding:

$$\begin{aligned} 4x^2 + 12x + 9 &\geq x^2 - 2x + 1 \\ 3x^2 + 14x + 8 &\geq 0 \end{aligned}$$

Factor:

$$(3x+2)(x+4) \geq 0$$

Critical points are $x = -4$ and $x = -\frac{2}{3}$. Since the quadratic opens upward, the inequality holds outside the interval between the roots:

$$x \leq -4 \quad \text{or} \quad x \geq -\frac{2}{3}$$

We must also exclude $x = 1$, where the denominator is zero.

Therefore, the domain of $f(x)$ is:

$$\boxed{(-\infty, -4] \cup [-\frac{2}{3}, 1) \cup (1, \infty)}$$

(b) Derivative:

Using the derivative formula:

$$\frac{d}{dx} (\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Let

$$u = \frac{2x+3}{x-1}$$

Then

$$\frac{du}{dx} = \frac{(x-1)(2) - (2x+3)(1)}{(x-1)^2} = \frac{-5}{(x-1)^2}$$

Substitute into the formula:

$$f'(x) = \frac{1}{\left| \frac{2x+3}{x-1} \right| \sqrt{\left(\frac{2x+3}{x-1} \right)^2 - 1}} \cdot \frac{-5}{(x-1)^2}$$

$$\boxed{f'(x) = \frac{-5}{(x-1)^2 \left| \frac{2x+3}{x-1} \right| \sqrt{\left(\frac{2x+3}{x-1} \right)^2 - 1}}}$$

Question 6

Let

$$f(x) = \tan^{-1} \left(\sqrt{1 - x^2} \right)$$

- (a) Find the derivative $f'(x)$.
- (b) Determine the domain of $f'(x)$.

Question 6 - Solution

We are given:

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

(a) Finding $f'(x)$:

Let $u(x) = \sqrt{1-x^2}$, so:

$$f(x) = \tan^{-1}(u) \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot u'(x)$$

First, compute $u'(x)$:

$$u(x) = (1-x^2)^{1/2} \Rightarrow u'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Now compute:

$$f'(x) = \frac{1}{1+(1-x^2)} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}}$$

Final Answer:

$$f'(x) = \frac{-x}{(2-x^2)\sqrt{1-x^2}}$$

(b) Domain of $f'(x)$:

We need: $\sqrt{1-x^2}$ to be real $1-x^2 \geq 0 \Rightarrow x \in [-1, 1]$, Denominator $\sqrt{1-x^2} \neq 0$ exclude $x = \pm 1$,
Also ensure $2-x^2 \neq 0 \Rightarrow x \neq \sqrt{2}, -\sqrt{2}$ (but these are already excluded)

So domain of $f'(x)$ is:

$$(-1, 1)$$

Question 7

Let

$$f(x) = \arcsin\left(\frac{2x}{\sqrt{1+x^2}+1}\right).$$

- (a) Determine the real domain.
- (b) Decide whether $f(x) = \arctan x$ holds on that domain. Justify your answer.
- (c) Compute $f'(x)$ wherever it exists.

Question 7 - Solution

Domain. Let $s = \sqrt{1+x^2} \geq 1$. The argument of arcsin must satisfy

$$\frac{4x^2}{(s+1)^2} = \frac{4(s-1)}{s+1} \leq 1.$$

This is equivalent to $s \leq 5/3$, hence

$$\boxed{D = [-4/3, 4/3]}.$$

Test the claimed identity. At $x = 4/3$, the arcsine is $\pi/2$, while $\arctan(4/3) < \pi/2$. Thus $f(x) = \arctan x$ is false.

Differentiate. For $|x| < 4/3$, put $u = 2x/(s+1)$. Then

$$u' = \frac{2}{s(s+1)}, \quad 1 - u^2 = \frac{5-3s}{s+1}.$$

Consequently

$$\boxed{f'(x) = \frac{2}{s\sqrt{s+1}\sqrt{5-3s}}}, \quad s = \sqrt{1+x^2}.$$

The derivative is defined on $(-4/3, 4/3)$ and grows without bound toward either endpoint.

Question 8

Let $f(x) = \tan^{-1}(\sqrt{1-x^2})$, where $-1 < x < 1$.

- (a) Compute $f'(x)$.
- (b) Simplify your result as much as possible.

Question 8 - Solution

We are given:

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

(a) Differentiate using the chain rule:

Let $u = \sqrt{1-x^2}$, so:

$$f(x) = \tan^{-1}(u) \Rightarrow f'(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

Now compute each part:

$$u = (1-x^2)^{1/2}, \quad \frac{du}{dx} = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Now plug into the derivative:

$$f'(x) = \frac{1}{1+(1-x^2)} \cdot \frac{-x}{\sqrt{1-x^2}} = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}}$$

$$f'(x) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

(b) Final simplified form:

$$f'(x) = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$

Question 9

Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, where $-1 < x < 1$.

- (a) Compute $f'(x)$.
- (b) Show that the result simplifies to a rational function.

Question 9 - Solution

We are given:

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let $u = \frac{2x}{1+x^2}$, so:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

Step 1: Differentiate $u = \frac{2x}{1+x^2}$:

Using the quotient rule:

$$u' = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

Step 2: Simplify $\sqrt{1-u^2}$:

$$\begin{aligned} u = \frac{2x}{1+x^2} &\Rightarrow u^2 = \frac{4x^2}{(1+x^2)^2} \Rightarrow 1-u^2 = 1 - \frac{4x^2}{(1+x^2)^2} = \frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2} \\ &= \frac{1+2x^2+x^4-4x^2}{(1+x^2)^2} = \frac{1-2x^2+x^4}{(1+x^2)^2} = \frac{(1-x^2)^2}{(1+x^2)^2} \Rightarrow \sqrt{1-u^2} = \frac{|1-x^2|}{1+x^2} \end{aligned}$$

Since $-1 < x < 1$, we have $1-x^2 > 0$, so:

$$\sqrt{1-u^2} = \frac{1-x^2}{1+x^2}$$

Step 3: Combine all parts:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u' = \frac{1}{\frac{1-x^2}{1+x^2}} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = \frac{1+x^2}{1-x^2} \cdot \frac{2(1-x^2)}{(1+x^2)^2}$$

Cancel $1-x^2$ and one factor of $1+x^2$:

$$f'(x) = \frac{2}{1+x^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{2}{1+x^2}}$$

Question 10

Let $f(x) = \tan^{-1}(\sqrt{1-x^2})$ for $-1 < x < 1$.

- (a) Find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \tan^{-1}(\sqrt{1-x^2})$$

Let $u = \sqrt{1-x^2}$, then by the chain rule:

$$f'(x) = \frac{1}{1+u^2} \cdot u'$$

Step 1: Compute u' :

$$u = (1-x^2)^{1/2} \Rightarrow u' = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

Step 2: Compute $1+u^2$:

$$1+u^2 = 1+(1-x^2) = 2-x^2$$

Step 3: Combine everything:

$$f'(x) = \frac{1}{1+u^2} \cdot u' = \frac{1}{2-x^2} \cdot \frac{-x}{\sqrt{1-x^2}} = \boxed{\frac{-x}{(2-x^2)\sqrt{1-x^2}}}$$