

Question 1

Let

$$f(x) = \ln(x^2 + 1) \cdot e^{3x}$$

- (a) Differentiate $f(x)$ using the product rule and the chain rule.
- (b) Simplify your answer.

Question 1 - Solution

We are given:

$$f(x) = \ln(x^2 + 1) \cdot e^{3x}$$

(a) Differentiate Using Product Rule:

Let:

$$u(x) = \ln(x^2 + 1), \quad v(x) = e^{3x}$$

Then:

$$u'(x) = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}, \quad v'(x) = 3e^{3x}$$

Now apply the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = \frac{2x}{x^2 + 1} \cdot e^{3x} + \ln(x^2 + 1) \cdot 3e^{3x}$$

(b) Final Simplified Answer:

Factor out e^{3x} :

$$f'(x) = e^{3x} \left(\frac{2x}{x^2 + 1} + 3 \ln(x^2 + 1) \right)$$

$$\boxed{f'(x) = e^{3x} \left(\frac{2x}{x^2 + 1} + 3 \ln(x^2 + 1) \right)}$$

Question 2

Consider the function:

$$f(x) = \frac{e^x}{\ln(x)}$$

- (a) Find the derivative $f'(x)$.
- (b) State the domain of $f(x)$ and $f'(x)$.

Question 2 - Solution

We are given:

$$f(x) = \frac{e^x}{\ln(x)}$$

(a) Differentiate $f(x)$:

This is a quotient, so we apply the quotient rule:

$$f'(x) = \frac{(\ln(x) \cdot \frac{d}{dx}[e^x]) - (e^x \cdot \frac{d}{dx}[\ln(x)])}{(\ln(x))^2}$$

Differentiate:

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

Now substitute:

$$f'(x) = \frac{\ln(x) \cdot e^x - e^x \cdot \frac{1}{x}}{(\ln(x))^2} = \frac{e^x (\ln(x) - \frac{1}{x})}{(\ln(x))^2}$$

$$f'(x) = \frac{e^x (\ln(x) - \frac{1}{x})}{(\ln(x))^2}$$

(b) Domain of $f(x)$:

$\ln(x)$ is only defined for $x > 0$

But we also need $\ln(x) \neq 0 \Rightarrow x \neq 1$

So domain of $f(x)$ is:

$$(0, 1) \cup (1, \infty)$$

The derivative involves the same terms (no new restrictions), so:

$$\text{Domain of } f'(x) = \boxed{(0, 1) \cup (1, \infty)}$$

Question 3

Let $f(x) = x^x$ for $x > 0$.

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Evaluate $f'(x)$ at $x = 2$.

Question 3 - Solution

We are given:

$$f(x) = x^x, \quad x > 0$$

(a) Logarithmic Differentiation

Take natural log on both sides:

$$\ln(f(x)) = \ln(x^x) = x \ln(x)$$

Differentiate both sides implicitly:

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}[x \ln(x)]$$

Differentiate the right-hand side using product rule:

$$\frac{d}{dx}[x \ln(x)] = \ln(x) + 1$$

So:

$$\frac{1}{f(x)} \cdot f'(x) = \ln(x) + 1 \Rightarrow f'(x) = f(x) \cdot (\ln(x) + 1)$$

Recall $f(x) = x^x$, so:

$$\boxed{f'(x) = x^x(\ln(x) + 1)}$$

(b) Evaluate at $x = 2$:

$$f'(2) = 2^2(\ln(2) + 1) = 4(\ln(2) + 1)$$

Exact Answer:

$$\boxed{f'(2) = 4(\ln(2) + 1)}$$

Question 4

Let the function $f(x) = e^{x \ln(\sin x)}$, for $0 < x < \pi$.

- (a) Simplify the expression $f(x)$.
- (b) Find the derivative $f'(x)$.

Question 4 - Solution

We are given:

$$f(x) = e^{x \ln(\sin x)}, \quad 0 < x < \pi$$

(a) Simplify the expression:

Recall the identity:

$$e^{x \ln(\sin x)} = (e^{\ln(\sin x)})^x = (\sin x)^x$$

So:

$$f(x) = (\sin x)^x$$

(b) Differentiate $f(x) = (\sin x)^x$ using logarithmic differentiation:

Take natural log on both sides:

$$\ln(f(x)) = x \ln(\sin x)$$

Differentiate both sides:

$$\frac{1}{f(x)} \cdot f'(x) = \frac{d}{dx}[x \ln(\sin x)]$$

Apply product rule on the right-hand side:

$$\frac{d}{dx}[x \ln(\sin x)] = \ln(\sin x) + x \cdot \frac{\cos x}{\sin x} = \ln(\sin x) + x \cot x$$

Multiply both sides by $f(x) = (\sin x)^x$:

$$f'(x) = (\sin x)^x \cdot (\ln(\sin x) + x \cot x)$$

Final Answer:

$$\boxed{f'(x) = (\sin x)^x (\ln(\sin x) + x \cot x)}$$

Question 5

Let the function $f(x) = \ln\left(\frac{x^2\sqrt{e^x+1}}{\ln x}\right)$, for $x > 1$.

- (a) Simplify the expression using logarithmic properties.
- (b) Find the derivative $f'(x)$.

Question 5 - Solution

We are given:

$$f(x) = \ln \left(\frac{x^2 \sqrt{e^x + 1}}{\ln x} \right), \quad x > 1$$

(a) Simplify the expression:

Use logarithmic identities:

$$\ln \left(\frac{A \cdot B}{C} \right) = \ln(A) + \ln(B) - \ln(C)$$

Apply to this expression:

$$f(x) = \ln(x^2) + \ln \left((e^x + 1)^{1/2} \right) - \ln(\ln x)$$

Now simplify powers:

$$f(x) = 2 \ln x + \frac{1}{2} \ln(e^x + 1) - \ln(\ln x)$$

(b) Differentiate $f(x)$:

Differentiate term-by-term:

$$\begin{aligned} \frac{d}{dx} [2 \ln x] &= \frac{2}{x} \\ \frac{d}{dx} \left[\frac{1}{2} \ln(e^x + 1) \right] &= \frac{1}{2} \cdot \frac{e^x}{e^x + 1} \\ \frac{d}{dx} [\ln(\ln x)] &= \frac{1}{x \ln x} \end{aligned}$$

Putting it all together:

$$f'(x) = \frac{2}{x} + \frac{1}{2} \cdot \frac{e^x}{e^x + 1} - \frac{1}{x \ln x}$$

Final Answer:

$f'(x) = \frac{2}{x} + \frac{e^x}{2(e^x + 1)} - \frac{1}{x \ln x}$
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Question 6

Let $f(x) = x^x$, defined for $x > 0$.

- (a) Find the derivative $f'(x)$.
- (b) Evaluate $f'(1)$.

Question 6 - Solution

We are given:

$$f(x) = x^x, \quad x > 0$$

This is a function where the base and exponent both depend on x . To differentiate it, we take logarithms on both sides.

Step 1: Take natural log of both sides:

Let $y = x^x$. Then,

$$\ln y = \ln(x^x) = x \ln x$$

Step 2: Differentiate both sides implicitly:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \ln x) = \ln x + 1$$

Step 3: Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

Final answer:

$$\boxed{f'(x) = x^x(\ln x + 1)}$$

(b) Evaluate $f'(1)$:

$$f'(1) = 1^1(\ln 1 + 1) = 1(0 + 1) = \boxed{1}$$

Question 7

Let $f(x) = \ln\left(\frac{e^x+1}{e^x-1}\right)$, for $x > 0$.

- (a) Find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 7 - Solution

We are given:

$$f(x) = \ln \left(\frac{e^x + 1}{e^x - 1} \right)$$

Step 1: Use the derivative of the natural log of a quotient:

$$\frac{d}{dx} \ln \left(\frac{u(x)}{v(x)} \right) = \frac{d}{dx} [\ln(u(x)) - \ln(v(x))] = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)}$$

Here, $u(x) = e^x + 1$, and $v(x) = e^x - 1$

So,

$$f'(x) = \frac{d}{dx} [\ln(e^x + 1) - \ln(e^x - 1)] = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1}$$

Step 2: Combine the two terms:

$$f'(x) = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1}$$

Get a common denominator:

$$f'(x) = e^x \left(\frac{1}{e^x + 1} - \frac{1}{e^x - 1} \right) = e^x \left(\frac{(e^x - 1) - (e^x + 1)}{(e^x + 1)(e^x - 1)} \right) = e^x \left(\frac{e^x - 1 - e^x - 1}{e^{2x} - 1} \right)$$

$$f'(x) = e^x \left(\frac{-2}{e^{2x} - 1} \right) = \boxed{\frac{-2e^x}{e^{2x} - 1}}$$

Question 8

Let $f(x) = x^x$, where $x > 0$.

- (a) Find the derivative $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 8 - Solution

We are given:

$$f(x) = x^x \quad \text{for } x > 0$$

This is a function where the variable is both the base and the exponent. To differentiate it, we take the natural logarithm of both sides:

Let

$$y = x^x$$

Take the natural log of both sides:

$$\ln y = \ln(x^x) = x \ln x$$

Differentiate both sides implicitly:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \ln x)$$

Now differentiate the right-hand side using the product rule:

$$\frac{d}{dx}(x \ln x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y(\ln x + 1)$$

Recall that $y = x^x$, so:

$$f'(x) = \boxed{x^x(\ln x + 1)}$$

Question 9

Let $a > 0$ be a real constant and define the function

$$f(x) = \ln\left(\frac{e^{ax} + 1}{e^{ax} - 1}\right).$$

- (a) Find the derivative $f'(x)$ in terms of a .
- (b) Simplify your answer as much as possible.
- (c) Determine the domain of $f(x)$.
- (d) Show that $f'(x)$ can be written in the form

$$f'(x) = -\frac{2a}{e^{ax} - e^{-ax}}.$$

- (e) Hence, or otherwise, determine whether $f(x)$ is increasing or decreasing on its domain.

Question 9 - Solution

We are given

$$f(x) = \ln\left(\frac{e^{ax} + 1}{e^{ax} - 1}\right), \quad a > 0.$$

(a) Differentiation

Rewrite the function using logarithmic identities:

$$f(x) = \ln(e^{ax} + 1) - \ln(e^{ax} - 1).$$

Differentiate term by term using the chain rule:

$$f'(x) = \frac{ae^{ax}}{e^{ax} + 1} - \frac{ae^{ax}}{e^{ax} - 1}.$$

(b) Simplification

Factor out ae^{ax} :

$$f'(x) = ae^{ax} \left(\frac{1}{e^{ax} + 1} - \frac{1}{e^{ax} - 1} \right).$$

Combine the fractions:

$$= ae^{ax} \left(\frac{(e^{ax} - 1) - (e^{ax} + 1)}{(e^{ax} + 1)(e^{ax} - 1)} \right) = ae^{ax} \left(\frac{-2}{e^{2ax} - 1} \right).$$

Thus,

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax} - 1}.$$

(c) Domain of $f(x)$

The logarithm requires:

$$\frac{e^{ax} + 1}{e^{ax} - 1} > 0, \quad e^{ax} - 1 \neq 0.$$

Since $e^{ax} > 0$ for all x , the expression is positive only when

$$e^{ax} - 1 > 0 \quad \Rightarrow \quad x > 0.$$

Hence, the domain is:

$$\boxed{(0, \infty)}.$$

(d) Alternative Form

Multiply numerator and denominator of the derivative by e^{-ax} :

$$f'(x) = -\frac{2ae^{ax}}{e^{2ax} - 1} = -\frac{2a}{e^{ax} - e^{-ax}}.$$

(e) Monotonicity

For $x > 0$,

$$e^{ax} - e^{-ax} > 0, \quad a > 0.$$

Therefore,

$$f'(x) < 0 \quad \text{for all } x > 0.$$

The function $f(x)$ is strictly decreasing on its domain.

Question 10

Let $a > 0$ be a real constant and define

$$f(x) = x^{ax}, \quad x > 0.$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your answer as much as possible.
- (c) Find the second derivative $f''(x)$.
- (d) Determine all stationary points of $f(x)$.
- (e) Hence, determine the intervals on which $f(x)$ is increasing and decreasing.

Question 10 - Solution

We are given

$$f(x) = x^{ax}, \quad x > 0, \quad a > 0.$$

(a) Logarithmic Differentiation

Let

$$y = x^{ax}.$$

Taking natural logarithms:

$$\ln y = ax \ln x.$$

(b) First Derivative

Differentiate both sides with respect to x :

$$\frac{1}{y} \frac{dy}{dx} = a \ln x + a.$$

Multiply through by y :

$$\frac{dy}{dx} = y(a \ln x + a).$$

Substitute back $y = x^{ax}$:

$$f'(x) = x^{ax} a(\ln x + 1).$$

$$\boxed{f'(x) = ax^{ax}(\ln x + 1)}$$

(c) Second Derivative

Differentiate $f'(x)$ using the product rule:

$$f'(x) = ax^{ax}(\ln x + 1).$$

Let

$$u = x^{ax}, \quad v = \ln x + 1.$$

Then:

$$u' = ax^{ax}(\ln x + 1), \quad v' = \frac{1}{x}.$$

Thus,

$$f''(x) = a[u'v + uv'] = a \left[x^{ax} a(\ln x + 1)^2 + x^{ax} \frac{1}{x} \right].$$

Factor out x^{ax} :

$$f''(x) = ax^{ax} \left[a(\ln x + 1)^2 + \frac{1}{x} \right].$$

(d) Stationary Points

Stationary points occur when

$$f'(x) = 0.$$

Since $a > 0$ and $x^{ax} > 0$ for $x > 0$, we require:

$$\ln x + 1 = 0.$$

Solving:

$$\ln x = -1 \quad \Rightarrow \quad x = e^{-1}.$$

Thus, the only stationary point occurs at:

$$\boxed{x = \frac{1}{e}}.$$

(e) Increasing and Decreasing Behaviour

From

$$f'(x) = ax^{ax}(\ln x + 1),$$

the sign of $f'(x)$ depends on $\ln x + 1$.

- If $0 < x < \frac{1}{e}$, then $\ln x + 1 < 0 \Rightarrow f'(x) < 0$.
- If $x > \frac{1}{e}$, then $\ln x + 1 > 0 \Rightarrow f'(x) > 0$.

Decreasing on $(0, e^{-1})$, Increasing on (e^{-1}, ∞) .
