

Question 1

Let

$$f(x) = \sin(x) \cos(x) + \tan(x)$$

- (a) Find $f'(x)$, the derivative of $f(x)$, using standard derivative rules.
- (b) Simplify the result as much as possible.

Question 1 - Solution

We are given:

$$f(x) = \sin(x) \cos(x) + \tan(x)$$

(a) Differentiate each term:

We apply the product rule to the first term $\sin(x) \cos(x)$, and recall the derivative of $\tan(x)$ is $\sec^2(x)$:

$$\frac{d}{dx} [\sin(x) \cos(x)] = \cos(x) \cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

So the derivative of the entire function is:

$$f'(x) = \cos^2(x) - \sin^2(x) + \sec^2(x)$$

(b) Simplify the result:

Note that:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

Thus:

$$\boxed{f'(x) = \cos(2x) + \sec^2(x)}$$

Question 2

Let

$$f(x) = x^2 \sec(x) + x \cot(x)$$

- (a) Compute the derivative $f'(x)$.
- (b) Simplify your result as much as possible.

Question 2 - Solution

We are given:

$$f(x) = x^2 \sec(x) + x \cot(x)$$

(a) Differentiate each term:

We apply the product rule to both terms:

First Term: $x^2 \sec(x)$

Let $u = x^2$, $v = \sec(x)$, then:

$$\frac{d}{dx}[x^2 \sec(x)] = \frac{d}{dx}[u \cdot v] = u'v + uv' = 2x \sec(x) + x^2 \sec(x) \tan(x)$$

Second Term: $x \cot(x)$

Let $u = x$, $v = \cot(x)$, then:

$$\frac{d}{dx}[x \cot(x)] = u'v + uv' = 1 \cdot \cot(x) + x(-\csc^2(x)) = \cot(x) - x \csc^2(x)$$

(b) Combine all terms:

$$f'(x) = 2x \sec(x) + x^2 \sec(x) \tan(x) + \cot(x) - x \csc^2(x)$$

Final Answer:

$f'(x) = 2x \sec(x) + x^2 \sec(x) \tan(x) + \cot(x) - x \csc^2(x)$
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Question 3

Let

$$f(x) = \sin(x) \cos(x) + \tan(x) \sec(x)$$

- (a) Find $f'(x)$.
- (b) Simplify the expression.

Question 3 - Solution

We are given:

$$f(x) = \sin(x) \cos(x) + \tan(x) \sec(x)$$

(a) Differentiate each term:

Term 1: $\sin(x) \cos(x)$

Using the product rule:

$$\frac{d}{dx}[\sin(x) \cos(x)] = \cos(x) \cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

Term 2: $\tan(x) \sec(x)$

Using the product rule:

$$\frac{d}{dx}[\tan(x) \sec(x)] = \sec^2(x) \sec(x) + \tan(x) \sec(x) \tan(x) = \sec^3(x) + \sec(x) \tan^2(x)$$

(b) Combine and simplify:

$$f'(x) = \cos^2(x) - \sin^2(x) + \sec^3(x) + \sec(x) \tan^2(x)$$

Use trigonometric identities:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

$$\tan^2(x) = \sec^2(x) - 1$$

$$\sec(x) \tan^2(x) = \sec^3(x) - \sec(x)$$

Substitute and combine like terms:

$$f'(x) = \cos(2x) + 2 \sec^3(x) - \sec(x)$$

Final Answer:

$f'(x) = \cos(2x) + 2 \sec^3(x) - \sec(x)$
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Question 4

Let

$$f(x) = \cot(x) \csc(x) + \cos^2(x)$$

- (a) Find the derivative $f'(x)$.
- (b) Simplify the expression as much as possible.

Question 4 - Solution

We are given:

$$f(x) = \cot(x) \csc(x) + \cos^2(x)$$

(a) Differentiate each term separately.

Term 1: $\cot(x) \csc(x)$

Using the product rule:

$$\begin{aligned}\frac{d}{dx}[\cot(x) \csc(x)] &= \cot(x)(-\csc(x)) + \csc(x)(-\csc^2(x)) \\ &= -\cot^2(x) \csc(x) - \csc^3(x)\end{aligned}$$

Term 2: $\cos^2(x)$

Using the chain rule:

$$\frac{d}{dx}[\cos^2(x)] = 2 \cos(x)(-\sin(x)) = -2 \sin(x) \cos(x)$$

(b) Combine and simplify:

$$f'(x) = -\cot^2(x) \csc(x) - \csc^3(x) - 2 \sin(x) \cos(x)$$

Use the identities:

$$\cot^2(x) = \csc^2(x) - 1 \quad 2 \sin(x) \cos(x) = \sin(2x)$$

$$-\cot^2(x) \csc(x) = -(\csc^2(x) - 1) \csc(x) = -\csc^3(x) + \csc(x)$$

Substitute and combine like terms:

$$f'(x) = \csc(x) - 2 \csc^3(x) - \sin(2x)$$

Final Answer:

$$\boxed{f'(x) = \csc(x) - 2 \csc^3(x) - \sin(2x)}$$

Question 5

Let

$$f(x) = x \tan(x) - \sec(x)$$

- (a) Find the derivative $f'(x)$.
- (b) Identify any values of x where $f'(x)$ is undefined.

Question 5 - Solution

We are given:

$$f(x) = x \tan(x) - \sec(x)$$

(a) Differentiate each term.

Term 1: $x \tan(x)$ Use the product rule:

$$\frac{d}{dx}[x \tan(x)] = x \cdot \sec^2(x) + \tan(x)$$

Term 2: $-\sec(x)$

$$\frac{d}{dx}[-\sec(x)] = -\sec(x) \tan(x)$$

Combine the terms:

$$f'(x) = x \sec^2(x) + \tan(x) - \sec(x) \tan(x)$$

Final Answer:

$$f'(x) = x \sec^2(x) + \tan(x) - \sec(x) \tan(x)$$

(b) Values where $f'(x)$ is undefined:

Note that $\tan(x)$ and $\sec(x)$ are undefined at:

$$x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

So, the derivative $f'(x)$ is undefined at:

$$x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

Question 6

Let

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

- (a) Find the derivative $f'(x)$.
- (b) Simplify the derivative as much as possible.

Question 6 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

(a) Use the Quotient Rule:

The quotient rule is:

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

Let:

$$g(x) = \sin(x), \quad h(x) = 1 + \cos(x)$$

$$g'(x) = \cos(x), \quad h'(x) = -\sin(x)$$

Then:

$$\begin{aligned} f'(x) &= \frac{\cos(x)(1 + \cos(x)) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} \\ &= \frac{\cos(x)(1 + \cos(x)) + \sin^2(x)}{(1 + \cos(x))^2} \end{aligned}$$

(b) Simplify the numerator:

First expand:

$$\cos(x)(1 + \cos(x)) = \cos(x) + \cos^2(x)$$

So the full numerator is:

$$\cos(x) + \cos^2(x) + \sin^2(x)$$

Use the identity:

$$\cos^2(x) + \sin^2(x) = 1$$

So the numerator becomes:

$$\cos(x) + 1$$

Therefore,

$$f'(x) = \frac{\cos(x) + 1}{(1 + \cos(x))^2}$$

Simplify:

$$f'(x) = \frac{1}{1 + \cos(x)}$$

Final Answer:

$$\boxed{f'(x) = \frac{1}{1 + \cos(x)}}$$

Question 7

Let

$$f(x) = \tan(x) \cdot \sec(x)$$

- (a) Find the derivative $f'(x)$.
- (b) Simplify the result using trigonometric identities.

Question 7 - Solution

We are given:

$$f(x) = \tan(x) \cdot \sec(x)$$

(a) Apply the Product Rule:

The product rule states:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Let:

$$u(x) = \tan(x), \quad v(x) = \sec(x)$$

$$u'(x) = \sec^2(x), \quad v'(x) = \sec(x) \tan(x)$$

Then:

$$\begin{aligned} f'(x) &= \sec^2(x) \cdot \sec(x) + \tan(x) \cdot \sec(x) \cdot \tan(x) \\ &= \sec^3(x) + \sec(x) \tan^2(x) \end{aligned}$$

(b) Factor the result:

Both terms have a factor of $\sec(x)$, so factor it out:

$$f'(x) = \sec(x) (\sec^2(x) + \tan^2(x))$$

Final Answer:

$$\boxed{f'(x) = \sec(x) (\sec^2(x) + \tan^2(x))}$$

Question 8

Let

$$f(x) = \frac{\sin(x) \cos(x)}{1 + \sin^2(x)}$$

- (a) Differentiate $f(x)$.
- (b) Simplify the result as much as possible.

Question 8 - Solution

We are given:

$$f(x) = \frac{\sin(x) \cos(x)}{1 + \sin^2(x)}$$

Step 1: Rewrite using a double-angle identity.

Recall:

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

So we rewrite the function:

$$f(x) = \frac{\frac{1}{2} \sin(2x)}{1 + \sin^2(x)}$$

Step 2: Apply the quotient rule.

Let:

$$u(x) = \frac{1}{2} \sin(2x), \quad v(x) = 1 + \sin^2(x)$$

Differentiate each function:

$$u'(x) = \cos(2x)$$

$$v'(x) = 2 \sin(x) \cos(x) = \sin(2x)$$

The quotient rule states:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Substitute:

$$f'(x) = \frac{\cos(2x)(1 + \sin^2(x)) - \frac{1}{2} \sin(2x) \sin(2x)}{(1 + \sin^2(x))^2}$$

Step 3: Simplify carefully.

Rewrite the squared sine:

$$\sin^2(2x) = 1 - \cos^2(2x)$$

So the numerator becomes:

$$\cos(2x)(1 + \sin^2(x)) - \frac{1}{2}(1 - \cos^2(2x))$$

Expand:

$$= \cos(2x) + \cos(2x) \sin^2(x) - \frac{1}{2} + \frac{1}{2} \cos^2(2x)$$

Now use:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Substitute:

$$\cos(2x) \sin^2(x) = \frac{1}{2} \cos(2x) - \frac{1}{2} \cos^2(2x)$$

Combine terms:

$$= \frac{3}{2} \cos(2x) - \frac{1}{2}$$

Final Simplified Result:

$$f'(x) = \frac{3 \cos(2x) - 1}{2(1 + \sin^2(x))^2}$$

Final Answer:

$$f'(x) = \frac{3 \cos(2x) - 1}{2(1 + \sin^2(x))^2}$$

Question 9

Let

$$f(x) = x \tan(x)$$

- (a) Use the product rule to find the derivative of $f(x)$.
- (b) Find the equation of the tangent line to $f(x)$ at $x = \frac{\pi}{4}$.

Question 9 - Solution

We are given:

$$f(x) = x \tan(x)$$

(a) Differentiate Using the Product Rule:

Let:

$$u(x) = x, \quad v(x) = \tan(x)$$

Then:

$$u'(x) = 1, \quad v'(x) = \sec^2(x)$$

Product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = 1 \cdot \tan(x) + x \cdot \sec^2(x) = \tan(x) + x \sec^2(x)$$

$$\boxed{f'(x) = \tan(x) + x \sec^2(x)}$$

(b) Tangent Line at $x = \frac{\pi}{4}$:

Evaluate:

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot \tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot 1 = \frac{\pi}{4}$$

$$f'\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \cdot \sec^2\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{4} \cdot 2 = 1 + \frac{\pi}{2}$$

Use point-slope form:

$$y - f\left(\frac{\pi}{4}\right) = f'\left(\frac{\pi}{4}\right) \left(x - \frac{\pi}{4}\right) \Rightarrow y - \frac{\pi}{4} = \left(1 + \frac{\pi}{2}\right) \left(x - \frac{\pi}{4}\right)$$

Final Answer: The tangent line is:

$$\boxed{y = \left(1 + \frac{\pi}{2}\right) \left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}}$$

Question 10

Consider the function:

$$f(x) = \sin(x) \cos(x)$$

- (a) Differentiate $f(x)$ using the product rule.
- (b) Simplify the derivative using trigonometric identities.
- (c) Evaluate $f'(x)$ at $x = \frac{\pi}{3}$.

Question 10 - Solution

We are given:

$$f(x) = \sin(x) \cos(x)$$

(a) Differentiate Using the Product Rule:

Let:

$$u(x) = \sin(x), \quad v(x) = \cos(x)$$

Then:

$$u'(x) = \cos(x), \quad v'(x) = -\sin(x)$$

Using the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x) = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

$$\boxed{f'(x) = \cos^2(x) - \sin^2(x)}$$

(b) Simplify Using Trig Identity:

We use the identity:

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

So:

$$\boxed{f'(x) = \cos(2x)}$$

(c) Evaluate at $x = \frac{\pi}{3}$:

$$f'\left(\frac{\pi}{3}\right) = \cos\left(2 \cdot \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\boxed{f'\left(\frac{\pi}{3}\right) = -\frac{1}{2}}$$