

Question 1

Let

$$f(x) = x^2 \cdot \ln(x)$$

- (a) Differentiate $f(x)$ using the product rule.
- (b) Find the equation of the tangent line to the graph of $f(x)$ at $x = 1$.

Question 1 - Solution

We are given:

$$f(x) = x^2 \cdot \ln(x)$$

Let:

$$u = x^2, \quad v = \ln(x) \Rightarrow u' = 2x, \quad v' = \frac{1}{x}$$

Apply the product rule:

$$f'(x) = u'v + uv' = 2x \ln(x) + x^2 \cdot \frac{1}{x} = 2x \ln(x) + x$$

Final derivative:

$$f'(x) = 2x \ln(x) + x$$

Now evaluate the tangent line at $x = 1$:

$$f(1) = 1^2 \cdot \ln(1) = 0$$

$$f'(1) = 2(1) \ln(1) + 1 = 0 + 1 = 1$$

Point on the curve: $(1, 0)$ Slope of tangent: 1

Use point-slope form:

$$y - 0 = 1(x - 1) \Rightarrow y = x - 1$$

Equation of the tangent line:

$$y = x - 1$$

Question 2

Let

$$f(x) = \frac{x^2 \sin(x)}{x + 1}$$

- (a) Differentiate $f(x)$ using the quotient rule.
- (b) Simplify your answer as much as possible.

Question 2 - Solution

We are given:

$$f(x) = \frac{x^2 \sin(x)}{x+1}$$

Let

$$u(x) = x^2 \sin(x), \quad v(x) = x+1$$

Then using the **quotient rule**:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

We compute:

$$u'(x) = \frac{d}{dx}[x^2 \sin(x)] = 2x \sin(x) + x^2 \cos(x) \quad (\text{product rule})$$

$$v'(x) = \frac{d}{dx}[x+1] = 1$$

Now plug into the quotient rule:

$$f'(x) = \frac{(x+1)(2x \sin(x) + x^2 \cos(x)) - x^2 \sin(x) \cdot 1}{(x+1)^2}$$

Now expand and simplify the numerator:

$$= \frac{(x+1)(2x \sin(x) + x^2 \cos(x)) - x^2 \sin(x)}{(x+1)^2}$$

Distribute:

$$= \frac{2x(x+1) \sin(x) + x^2(x+1) \cos(x) - x^2 \sin(x)}{(x+1)^2}$$

Final simplified expression:

$$f'(x) = \frac{2x(x+1) \sin(x) - x^2 \sin(x) + x^2(x+1) \cos(x)}{(x+1)^2}$$

Question 3

Let

$$f(x) = \frac{\ln(x)}{x \cos(x)}$$

- (a) Differentiate $f(x)$ using the quotient and product rules.
- (b) Simplify your answer as much as possible.

Question 3 - Solution

We are given:

$$f(x) = \frac{\ln(x)}{x \cos(x)}$$

Let

$$u(x) = \ln(x), \quad v(x) = x \cos(x)$$

Then by the **quotient rule**:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Compute each derivative:

$$u'(x) = \frac{1}{x}$$

For $v(x) = x \cos(x)$, use the **product rule**:

$$v'(x) = \frac{d}{dx}[x \cos(x)] = \cos(x) - x \sin(x)$$

Now substitute into the quotient rule:

$$f'(x) = \frac{x \cos(x) \cdot \frac{1}{x} - \ln(x)(\cos(x) - x \sin(x))}{(x \cos(x))^2}$$

Simplify numerator:

$$f'(x) = \frac{\cos(x) - \ln(x) \cos(x) + \ln(x)x \sin(x)}{x^2 \cos^2(x)}$$

Final Answer:

$$f'(x) = \frac{\cos(x)(1 - \ln(x)) + \ln(x)x \sin(x)}{x^2 \cos^2(x)}$$

Question 4

Let

$$f(x) = (x^2 + 3x) \cdot e^x$$

- (a) Use the product rule to differentiate $f(x)$.
- (b) Factor and simplify your final expression.

Question 4 - Solution

We are given:

$$f(x) = (x^2 + 3x) \cdot e^x$$

Let:

$$u(x) = x^2 + 3x, \quad v(x) = e^x$$

By the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Differentiate each part:

$$u'(x) = 2x + 3, \quad v'(x) = e^x$$

Now plug into the formula:

$$f'(x) = (2x + 3)e^x + (x^2 + 3x)e^x$$

Factor out e^x :

$$f'(x) = e^x [(2x + 3) + (x^2 + 3x)]$$

Simplify inside the brackets:

$$(2x + 3) + (x^2 + 3x) = x^2 + 5x + 3$$

Final Answer:

$$\boxed{f'(x) = e^x(x^2 + 5x + 3)}$$

Question 5

Let

$$f(x) = x^2 e^x \cos x$$

- (a) Compute $f'(x)$ using appropriate differentiation rules.
- (b) Factor the result where possible.

Question 5 - Solution

We are given:

$$f(x) = x^2 e^x \cos x$$

This is a product of three functions: - $u(x) = x^2$ - $v(x) = e^x$ - $w(x) = \cos x$

Use the **product rule** for three functions:

$$f'(x) = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$$

Compute derivatives: - $u'(x) = 2x$ - $v'(x) = e^x$ - $w'(x) = -\sin x$

Now substitute:

$$f'(x) = 2xe^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x$$

Factor where possible:

$$f'(x) = e^x (\cos x (2x + x^2) - x^2 \sin x)$$

Final Answer:

$$\boxed{f'(x) = e^x [\cos x (x^2 + 2x) - x^2 \sin x]}$$

Question 6

Let

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

- (a) Compute $f'(x)$.
- (b) Simplify and factor your answer as much as possible.

Question 6 - Solution

We are given:

$$f(x) = \frac{x^2 e^{2x} \sin x}{1 + x^3}$$

Let

$$u(x) = x^2 e^{2x} \sin x, \quad v(x) = 1 + x^3$$

Using the quotient rule:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate $u(x)$

The function $u(x)$ is a product of three functions:

$$u(x) = x^2 \cdot e^{2x} \cdot \sin x$$

Apply the product rule:

$$u'(x) = (x^2)' e^{2x} \sin x + x^2 (e^{2x})' \sin x + x^2 e^{2x} (\sin x)'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^{2x})' = 2e^{2x}, \quad (\sin x)' = \cos x$$

So:

$$u'(x) = 2x e^{2x} \sin x + 2x^2 e^{2x} \sin x + x^2 e^{2x} \cos x$$

Factor:

$$u'(x) = e^{2x} [\sin x (2x + 2x^2) + x^2 \cos x]$$

Step 2: Differentiate $v(x)$

$$v'(x) = 3x^2$$

Step 3: Apply the Quotient Rule

$$f'(x) = \frac{e^{2x} [\sin x (2x + 2x^2) + x^2 \cos x] (1 + x^3) - x^2 e^{2x} \sin x (3x^2)}{(1 + x^3)^2}$$

Factor common terms in the numerator:

$$f'(x) = \frac{e^{2x} x [(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x]}{(1 + x^3)^2}$$

Final Answer:

$$\boxed{f'(x) = \frac{e^{2x} x [(1 + x^3)(2 \sin x + 2x \sin x + x \cos x) - 3x^3 \sin x]}{(1 + x^3)^2}}$$

Question 7

Let

$$f(x) = \frac{x \sin x}{x^2 + 1}$$

- (a) Compute $f'(x)$ using the quotient rule.
- (b) Simplify the derivative fully.

Question 7 - Solution

We are given:

$$f(x) = \frac{x \sin x}{x^2 + 1}$$

Let:

$$u(x) = x \sin x, \quad v(x) = x^2 + 1$$

We apply the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

First, compute $u'(x)$ using the product rule:

$$u'(x) = \sin x + x \cos x$$

Next, compute $v'(x)$:

$$v'(x) = 2x$$

Now apply the quotient rule:

$$f'(x) = \frac{(\sin x + x \cos x)(x^2 + 1) - x \sin x(2x)}{(x^2 + 1)^2}$$

Expand the numerator:

$$(\sin x)(x^2 + 1) + x \cos x(x^2 + 1) - 2x^2 \sin x$$

Distribute:

$$x^2 \sin x + \sin x + x^3 \cos x + x \cos x - 2x^2 \sin x$$

Combine like terms:

$$-x^2 \sin x + \sin x + x^3 \cos x + x \cos x$$

Factor where possible:

$$\sin x(1 - x^2) + x \cos x(x^2 + 1)$$

Final Answer:

$$f'(x) = \frac{\sin x(1 - x^2) + x \cos x(x^2 + 1)}{(x^2 + 1)^2}$$

Question 8

Let

$$f(x) = \frac{e^x \cos x}{\ln(x)}$$

- (a) Compute $f'(x)$ using the quotient rule.
- (b) State the domain of $f'(x)$.

Question 8 - Solution

We are given:

$$f(x) = \frac{e^x \cos x}{\ln(x)}$$

Let: - $u(x) = e^x \cos x$ - $v(x) = \ln(x)$

We'll use the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate $u(x) = e^x \cos x$ using the **product rule**:

$$u'(x) = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$$

Step 2: Differentiate $v(x) = \ln(x)$:

$$v'(x) = \frac{1}{x}$$

Now plug into the quotient rule:

$$f'(x) = \frac{e^x(\cos x - \sin x) \cdot \ln x - e^x \cos x \cdot \frac{1}{x}}{(\ln x)^2}$$

Factor e^x from the numerator:

$$f'(x) = \frac{e^x \left[(\cos x - \sin x) \ln x - \frac{\cos x}{x} \right]}{(\ln x)^2}$$

Final Answer:

$$f'(x) = \frac{e^x \left[(\cos x - \sin x) \ln x - \frac{\cos x}{x} \right]}{(\ln x)^2}$$

Domain of $f'(x)$:

$\ln x$ is defined only for $x > 0$

Also, $\ln x \neq 0 \Rightarrow x \neq 1$

And denominator has $\ln(x)^2$, so $x = 1$ is excluded

The term $\frac{\cos x}{x}$ is defined for $x > 0$

Domain:

$$(0, 1) \cup (1, \infty)$$

Question 9

Let

$$f(x) = (x^2 + 1) \tan(x)$$

- (a) Differentiate $f(x)$ using the product rule.
- (b) Determine all x for which $f'(x)$ is undefined in the interval $(-\pi, \pi)$.

Question 9 - Solution

We are given:

$$f(x) = (x^2 + 1) \tan(x)$$

Let: $u(x) = x^2 + 1$ $v(x) = \tan(x)$

We'll use the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

Step 1: Differentiate $u(x)$:

$$u'(x) = 2x$$

Step 2: Differentiate $v(x) = \tan(x)$:

$$v'(x) = \sec^2(x)$$

Now apply the product rule:

$$f'(x) = 2x \tan(x) + (x^2 + 1) \sec^2(x)$$

Final Answer:

$$f'(x) = 2x \tan(x) + (x^2 + 1) \sec^2(x)$$

(b) Where is $f'(x)$ undefined on $(-\pi, \pi)$?

$\tan(x)$ and $\sec^2(x)$ are undefined where $\cos(x) = 0$

$\cos(x) = 0$ at $x = -\frac{\pi}{2}, \frac{\pi}{2}$ in the given interval

So, $f'(x)$ is undefined at:

$$x = -\frac{\pi}{2}, \frac{\pi}{2}$$

Question 10

Let

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

- (a) Differentiate $f(x)$ using appropriate differentiation rules.
- (b) Simplify and factor your answer as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 e^x \sin(x)}{x^2 + 1}$$

This is a quotient of two functions:

$$u(x) = x^2 e^x \sin(x), \quad v(x) = x^2 + 1$$

We apply the **quotient rule**:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Step 1: Differentiate the numerator

The numerator is a product of three functions:

$$u(x) = x^2 \cdot e^x \cdot \sin(x)$$

Using the product rule for three functions:

$$u'(x) = (x^2)' e^x \sin(x) + x^2 (e^x)' \sin(x) + x^2 e^x (\sin(x))'$$

Compute derivatives:

$$(x^2)' = 2x, \quad (e^x)' = e^x, \quad (\sin x)' = \cos x$$

Substitute:

$$u'(x) = 2x e^x \sin(x) + x^2 e^x \sin(x) + x^2 e^x \cos(x)$$

Factor:

$$u'(x) = e^x [\sin(x)(2x + x^2) + x^2 \cos(x)]$$

Step 2: Differentiate the denominator

$$v'(x) = 2x$$

Step 3: Apply the quotient rule

$$f'(x) = \frac{e^x [\sin(x)(2x + x^2) + x^2 \cos(x)] (x^2 + 1) - x^2 e^x \sin(x)(2x)}{(x^2 + 1)^2}$$

Step 4: Simplify

Factor out e^x :

$$f'(x) = \frac{e^x \{ (x^2 + 1) [\sin(x)(2x + x^2) + x^2 \cos(x)] - 2x^3 \sin(x) \}}{(x^2 + 1)^2}$$

This is a fully factored and simplified form.

Final Answer:

$$f'(x) = \frac{e^x [(x^2 + 1) (\sin(x)(2x + x^2) + x^2 \cos(x)) - 2x^3 \sin(x)]}{(x^2 + 1)^2}$$