

Question 1

Let

$$f(x) = \frac{\sin(x)}{x^2 + 1} + x \cos(x).$$

- (a) Find the derivative $f'(x)$ using standard differentiation rules.
- (b) Determine the slope of the tangent line to the graph of $f(x)$ at $x = 0$.

Question 1 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{x^2 + 1} + x \cos(x)$$

(a) Find $f'(x)$:

Differentiate term by term.

First term (quotient rule):

$$\frac{d}{dx} \left(\frac{\sin(x)}{x^2 + 1} \right) = \frac{(x^2 + 1) \cos(x) - \sin(x)(2x)}{(x^2 + 1)^2}$$

Second term (product rule):

$$\frac{d}{dx} (x \cos(x)) = \cos(x) - x \sin(x)$$

Putting both together:

$$f'(x) = \frac{(x^2 + 1) \cos(x) - 2x \sin(x)}{(x^2 + 1)^2} + \cos(x) - x \sin(x)$$

(b) Find the slope at $x = 0$:

Evaluate $f'(0)$.

$$\begin{aligned} f'(0) &= \frac{(0^2 + 1) \cos(0) - 2(0) \sin(0)}{(0^2 + 1)^2} + \cos(0) - 0 \cdot \sin(0) \\ &= \frac{1 \cdot 1 - 0}{1^2} + 1 - 0 \\ &= 1 + 1 = \boxed{2} \end{aligned}$$

So, the slope of the tangent line to the curve at $x = 0$ is $\boxed{2}$.

Question 2

Let

$$f(x) = \sqrt{x^2 + 3x} + \frac{1}{\sqrt{x^2 + 3x}}.$$

- (a) Compute $f'(x)$ using differentiation rules.
- (b) Determine all points where the derivative does not exist.

Question 2 - Solution

Domain. Set $u = x^2 + 3x = x(x + 3)$. The reciprocal square root requires $u > 0$, so

$$D = (-\infty, -3) \cup (0, \infty).$$

Differentiate.

$$f = u^{1/2} + u^{-1/2}, \quad \boxed{f'(x) = \frac{(2x+3)(x^2+3x-1)}{2(x^2+3x)^{3/2}}}, \quad x \in D.$$

The derivative exists at every point of the function's domain. On $[-3, 0]$, the function itself is undefined; these are not nondifferentiable points of its graph.

Question 3

Let

$$f(x) = \frac{e^x}{\cos x} + \ln(x^2 + 1).$$

- (a) Find $f'(x)$.
- (b) Determine the domain of $f'(x)$.

Question 3 - Solution

We are given:

$$f(x) = \frac{e^x}{\cos x} + \ln(x^2 + 1)$$

(a) Find $f'(x)$:

Differentiate each term separately.

For the first term $\frac{e^x}{\cos x}$, use the quotient rule:

$$\frac{d}{dx} \left(\frac{e^x}{\cos x} \right) = \frac{(\cos x)(e^x) - e^x(-\sin x)}{(\cos x)^2} = \frac{e^x(\cos x + \sin x)}{\cos^2 x}$$

For the second term $\ln(x^2 + 1)$, use the chain rule:

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Combine both:

$$f'(x) = \frac{e^x(\cos x + \sin x)}{\cos^2 x} + \frac{2x}{x^2 + 1}$$

(b) Domain of $f'(x)$:

We must exclude values that cause division by zero.

$\cos x = 0$ when $x = \frac{\pi}{2} + n\pi$, where $n \in \mathbb{Z}$, $x^2 + 1 > 0$ for all real x , so no restrictions there

Domain: All real x such that $\cos x \neq 0$

$$f'(x) = \frac{e^x(\cos x + \sin x)}{\cos^2 x} + \frac{2x}{x^2 + 1}, \quad x \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \right\}$$

Question 4

Let

$$f(x) = \frac{\arctan x}{1 + x^2} + x^5 e^x.$$

- (a) Compute $f'(x)$.
- (b) Write the equation for critical numbers and approximate one in $(-5, -4)$ to six decimal places.

Question 4 - Solution**(a) Differentiate**

The quotient and product rules give

$$f'(x) = \frac{1 - 2x \arctan x}{(1 + x^2)^2} + e^x x^4 (x + 5).$$

(b) Critical numbers

The derivative exists for every real x , since $1 + x^2 > 0$.

Thus critical numbers solve $f'(x) = 0$. Bisection on $(-5, -4)$ gives

$$x \approx -4.995520.$$

The derivative has opposite signs at the ends of this interval, so continuity guarantees a root there.

Question 5

Let

$$f(x) = \frac{\ln(x)}{x^3}$$

- (a) Find the derivative $f'(x)$ using the quotient rule.
- (b) Determine any local extrema of $f(x)$ for $x > 0$.

Question 5 - Solution

We are given:

$$f(x) = \frac{\ln(x)}{x^3}$$

(a) Find $f'(x)$:

Use the quotient rule:

$$f'(x) = \frac{x^3 \cdot \frac{1}{x} - \ln(x) \cdot 3x^2}{x^6} = \frac{x^2 - 3x^2 \ln(x)}{x^6}$$

Factor the numerator:

$$f'(x) = \frac{x^2(1 - 3\ln(x))}{x^6} = \frac{1 - 3\ln(x)}{x^4}$$

So,

$$f'(x) = \boxed{\frac{1 - 3\ln(x)}{x^4}}$$

(b) Local Extrema:

Critical points occur where $f'(x) = 0$:

$$\frac{1 - 3\ln(x)}{x^4} = 0 \Rightarrow 1 - 3\ln(x) = 0 \Rightarrow \ln(x) = \frac{1}{3} \Rightarrow x = e^{1/3}$$

Now determine if this is a maximum or minimum by checking the sign of $f'(x)$ around $x = e^{1/3}$:

For $x < e^{1/3}$, say $x = 1$: $f'(1) = \frac{1 - 3\ln(1)}{1^4} = 1 > 0$

For $x > e^{1/3}$, say $x = 2$: $f'(2) = \frac{1 - 3\ln(2)}{2^4} \approx \frac{1 - 3(0.693)}{16} \approx \frac{-1.079}{16} < 0$

So $f(x)$ has a local maximum at $x = e^{1/3}$.

Conclusion:

$f'(x) = \frac{1 - 3\ln(x)}{x^4}$, Local maximum at $x = \boxed{e^{1/3}}$

Question 6

Let

$$f(x) = \sqrt{x^2 + 4x + 5}$$

- (a) Differentiate $f(x)$ using the chain rule.
- (b) Find all values of x where the tangent line to $f(x)$ is horizontal.

Question 6 - Solution

We are given:

$$f(x) = \sqrt{x^2 + 4x + 5} = (x^2 + 4x + 5)^{1/2}$$

(a) Differentiate $f(x)$:

Using the chain rule:

$$f'(x) = \frac{1}{2}(x^2 + 4x + 5)^{-1/2} \cdot (2x + 4)$$

Simplify:

$$f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$$

So,

$$f'(x) = \boxed{\frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}}$$

(b) Horizontal Tangents:

The tangent is horizontal when $f'(x) = 0$. That occurs when the numerator is zero:

$$2x + 4 = 0 \Rightarrow x = -2$$

Check that the denominator is defined at $x = -2$:

$$\sqrt{(-2)^2 + 4(-2) + 5} = \sqrt{4 - 8 + 5} = \sqrt{1} = 1$$

So it's defined, and the tangent is horizontal at $x = -2$.

Conclusion:

- $f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$
- Horizontal tangent at $x = \boxed{-2}$

Question 7

Let

$$f(x) = \frac{e^x \sin x}{x^2 + 1}$$

- (a) Differentiate $f(x)$ using the quotient rule.
- (b) Evaluate $f'(0)$.

Question 7 - Solution

We are given:

$$f(x) = \frac{e^x \sin x}{x^2 + 1}$$

(a) Use the Quotient Rule:

The quotient rule is:

$$f(x) = \frac{u(x)}{v(x)} \Rightarrow f'(x) = \frac{u'v - uv'}{v^2}$$

Let:

$$u(x) = e^x \sin x, \quad v(x) = x^2 + 1$$

Differentiate $u(x)$ using the product rule:

$$u'(x) = \frac{d}{dx}[e^x \sin x] = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

Differentiate $v(x)$:

$$v'(x) = \frac{d}{dx}[x^2 + 1] = 2x$$

Now apply the quotient rule:

$$f'(x) = \frac{e^x(\sin x + \cos x)(x^2 + 1) - e^x \sin x \cdot 2x}{(x^2 + 1)^2}$$

Factor out e^x :

$$f'(x) = \frac{e^x [(\sin x + \cos x)(x^2 + 1) - 2x \sin x]}{(x^2 + 1)^2}$$

(b) Evaluate $f'(0)$:

Substitute $x = 0$:

Numerator:

$$e^0 [(\sin 0 + \cos 0)(0^2 + 1) - 2(0) \sin 0] = 1 \cdot (0 + 1)(1) = 1$$

Denominator:

$$(0^2 + 1)^2 = 1^2 = 1$$

So:

$$f'(0) = \boxed{1}$$

Question 8

Let

$$f(x) = \ln \left(\sqrt{x^2 + 3x + 2} \right)$$

- (a) Simplify the function before differentiating.
- (b) Find $f'(x)$.

Question 8 - Solution**(a) Simplify the Function:**

We start with:

$$f(x) = \ln\left(\sqrt{x^2 + 3x + 2}\right)$$

Since

$$\sqrt{x^2 + 3x + 2} = (x^2 + 3x + 2)^{1/2},$$

we can write:

$$f(x) = \ln\left((x^2 + 3x + 2)^{1/2}\right)$$

Using the logarithm rule

$$\ln(A^r) = r \ln(A),$$

we get:

$$f(x) = \frac{1}{2} \ln(x^2 + 3x + 2)$$

So the simplified function is:

$$f(x) = \frac{1}{2} \ln(x^2 + 3x + 2)$$

(b) Find $f'(x)$:

Now differentiate:

$$f(x) = \frac{1}{2} \ln(x^2 + 3x + 2)$$

Using

$$\frac{d}{dx} \ln(u) = \frac{u'}{u},$$

where

$$u = x^2 + 3x + 2$$

and

$$u' = 2x + 3,$$

we get:

$$f'(x) = \frac{1}{2} \cdot \frac{2x + 3}{x^2 + 3x + 2}$$

Therefore,

$$f'(x) = \frac{2x + 3}{2(x^2 + 3x + 2)}$$

Question 9

Let

$$f(x) = \tan^2(x) + \sec^2(x)$$

- (a) Differentiate $f(x)$ using the standard differentiation formulas for trigonometric functions.
- (b) Simplify your answer.

Question 9 - Solution**(a) Differentiate $f(x)$:**

We are given:

$$f(x) = \tan^2(x) + \sec^2(x)$$

We apply the chain rule and known derivatives:

$$\frac{d}{dx}[\tan^2(x)] = 2 \tan(x) \sec^2(x) \text{ (using chain rule)}$$

$$\frac{d}{dx}[\sec^2(x)] = 2 \sec^2(x) \tan(x) \text{ (using chain rule)}$$

So:

$$f'(x) = 2 \tan(x) \sec^2(x) + 2 \sec^2(x) \tan(x)$$

(b) Simplify:

Both terms are the same:

$$f'(x) = 4 \tan(x) \sec^2(x)$$

Final Answer:

$f'(x) = 4 \tan(x) \sec^2(x)$

Question 10

Let

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

- (a) Differentiate $f(x)$ using the quotient rule.
- (b) Simplify the result as much as possible.

Question 10 - Solution

We are given:

$$f(x) = \frac{\sin(x)}{1 + \cos(x)}$$

Let:

$$u = \sin(x), \quad v = 1 + \cos(x)$$

Then:

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

Compute the derivatives:

$$u' = \cos(x), \quad v' = -\sin(x)$$

Apply the quotient rule:

$$f'(x) = \frac{(1 + \cos(x)) \cos(x) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2}$$

Simplify numerator:

$$(1 + \cos(x)) \cos(x) + \sin^2(x) = \cos(x) + \cos^2(x) + \sin^2(x)$$

Use the identity $\sin^2(x) + \cos^2(x) = 1$:

$$= \cos(x) + 1$$

So:

$$f'(x) = \frac{\cos(x) + 1}{(1 + \cos(x))^2}$$

Cancel common factor:

$$f'(x) = \frac{1}{1 + \cos(x)}$$

Final Answer:

$f'(x) = \frac{1}{1 + \cos(x)}$
