

Question 1

The position of a particle moving along a straight line is given by

$$s(t) = t^3 - 6t^2 + 9t \quad \text{for } t \geq 0$$

- (a) Find the velocity and acceleration functions.
- (b) At what time(s) is the particle at rest?
- (c) Determine when the particle is speeding up and when it is slowing down.

Question 1 - Solution

We are given the position function:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Velocity and Acceleration:

Velocity is the first derivative of position:

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = 6t - 12$$

(b) Particle at Rest:

The particle is at rest when $v(t) = 0$:

$$3t^2 - 12t + 9 = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow (t - 1)(t - 3) = 0 \Rightarrow t = 1, 3$$

So, the particle is at rest at $t = \boxed{1}$ and $t = \boxed{3}$.

(c) Speeding Up / Slowing Down:

Speed increases when velocity and acceleration have the same sign; slows down when signs differ.

Critical points from previous steps:

$$v(t) = 3t^2 - 12t + 9, \quad a(t) = 6t - 12$$

Velocity is 0 at $t = 1, 3$, acceleration is 0 at $t = 2$

Test signs in intervals:

$t < 1$: $v(t) > 0$, $a(t) < 0 \rightarrow$ slowing down

$1 < t < 2$: $v(t) < 0$, $a(t) < 0 \rightarrow$ speeding up

$2 < t < 3$: $v(t) < 0$, $a(t) > 0 \rightarrow$ slowing down

$t > 3$: $v(t) > 0$, $a(t) > 0 \rightarrow$ speeding up

Conclusion:

- Speeding up: $t \in (1, 2) \cup (3, \infty)$
- Slowing down: $t \in (0, 1) \cup (2, 3)$

Question 2

A rock is thrown vertically upward from the top of a cliff. Its height in meters after t seconds is given by:

$$h(t) = -5t^2 + 20t + 50$$

- (a) What is the velocity of the rock after 2 seconds?
- (b) When does the rock reach its maximum height?
- (c) When does the rock hit the ground?
- (d) What is the velocity of the rock at the moment it hits the ground?

Question 2 - Solution

We are given:

$$h(t) = -5t^2 + 20t + 50$$

(a) Velocity after 2 seconds:

Velocity is the derivative of height:

$$v(t) = h'(t) = -10t + 20$$

So,

$$v(2) = -10(2) + 20 = -20 + 20 = \boxed{0 \text{ m/s}}$$

(b) Maximum Height:

Maximum height occurs when velocity is zero:

$$v(t) = -10t + 20 = 0 \Rightarrow t = 2 \text{ seconds}$$

(c) When does the rock hit the ground?

The rock hits the ground when $h(t) = 0$:

$$-5t^2 + 20t + 50 = 0 \Rightarrow t^2 - 4t - 10 = 0 \Rightarrow t = \frac{4 \pm \sqrt{16 + 40}}{2} = \frac{4 \pm \sqrt{56}}{2} = \frac{4 \pm 2\sqrt{14}}{2} = 2 \pm \sqrt{14}$$

Since time must be positive, we choose the positive root:

$$t = 2 + \sqrt{14} \approx \boxed{5.74 \text{ seconds}}$$

(d) Velocity at impact:

Plug into the velocity function:

$$v(2 + \sqrt{14}) = -10(2 + \sqrt{14}) + 20 = -20 - 10\sqrt{14} + 20 = -10\sqrt{14}$$

So the velocity at impact is:

$$\boxed{-10\sqrt{14} \text{ m/s}} \quad (\text{downward})$$

Question 3

The position of a particle moving along a straight line is given by:

$$s(t) = 3t^3 - 15t^2 + 18t \quad (\text{in meters}), \text{ for } t \geq 0.$$

- (a) Find the velocity and acceleration functions of the particle.
- (b) At what times is the particle at rest?
- (c) Determine the intervals when the particle is moving forward.
- (d) When is the acceleration zero, and what does it signify?

Question 3 - Solution

Given:

$$s(t) = 3t^3 - 15t^2 + 18t$$

(a) Velocity and Acceleration:

Velocity is the derivative of position:

$$v(t) = s'(t) = 9t^2 - 30t + 18$$

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = 18t - 30$$

(b) Particle at Rest:

Set velocity to zero:

$$9t^2 - 30t + 18 = 0 \Rightarrow t = \frac{30 \pm \sqrt{(-30)^2 - 4(9)(18)}}{2(9)} = \frac{30 \pm \sqrt{900 - 648}}{18} = \frac{30 \pm \sqrt{252}}{18}$$

$$\sqrt{252} = 6\sqrt{7}, \quad t = \frac{30 \pm 6\sqrt{7}}{18} = \frac{5 \pm \sqrt{7}}{3}$$

So the particle is at rest at:

$$t = \frac{5 \pm \sqrt{7}}{3}$$

(c) Particle Moving Forward:

Forward motion means $v(t) > 0$. Use sign chart for:

$$v(t) = 9t^2 - 30t + 18 = 9(t - t_1)(t - t_2), \quad \text{where } t_1 = \frac{5 - \sqrt{7}}{3}, t_2 = \frac{5 + \sqrt{7}}{3}$$

This quadratic opens upward (coefficient of t^2 is positive), so:

$$v(t) > 0 \quad \text{on } (0, t_1) \cup (t_2, \infty)$$

So the particle moves forward on:

$$\left(0, \frac{5 - \sqrt{7}}{3}\right) \cup \left(\frac{5 + \sqrt{7}}{3}, \infty\right)$$

(d) Acceleration Zero:

$$a(t) = 18t - 30 = 0 \Rightarrow t = \frac{30}{18} = \frac{5}{3}$$

This is when the particle changes concavity or when velocity changes most rapidly.

Conclusion:

$$a(t) = 0 \text{ when } t = \frac{5}{3} \text{ seconds}$$

Question 4

The cost (in dollars) to produce x items is given by the function:

$$C(x) = 50x + 300\sqrt{x}$$

- (a) Find the marginal cost function.
- (b) Interpret the meaning of $C'(100)$.
- (c) Approximate the cost of producing the 101st item using the derivative.
- (d) Compute the actual cost of producing the 101st item and compare with your approximation.

Question 4 - Solution

Marginal cost. For $x > 0$,

$$C'(x) = 50 + 150/\sqrt{x}.$$

Thus $C'(100) = 65$ dollars per item. This is the instantaneous cost rate and predicts a cost of approximately $\boxed{\$65}$ for one additional item.

Actual additional cost.

$$C(101) - C(100) = 50 + 300(\sqrt{101} - 10) \approx \boxed{\$64.96}.$$

The derivative estimate exceeds the actual cost by about \$0.04. This agrees with $C''(x) = -75x^{-3/2} < 0$: the marginal cost decreases over the next item.

Question 5

The population of a certain city (in thousands) $P(t)$, as a function of time t (in years since 2000), is modeled by the function:

$$P(t) = 500 + 40t - 0.5t^2$$

- (a) Find the rate of change of the population with respect to time.
- (b) What is the population growth rate in the year 2010?
- (c) At what year is the population growth rate zero?
- (d) Interpret your result from (c) in the context of the population.

Question 5 - Solution

We are given:

$$P(t) = 500 + 40t - 0.5t^2$$

(a) Rate of Change of Population:

The rate of change of the population is given by the derivative:

$$P'(t) = \frac{d}{dt}(500 + 40t - 0.5t^2) = 40 - t$$

$$P'(t) = 40 - t$$

(b) Growth Rate in 2010:

Since t is years since 2000, 2010 corresponds to $t = 10$.

$$P'(10) = 40 - 10 = 30$$

Interpretation: In 2010, the population was increasing at a rate of 30 thousand people per year.

$$P'(10) = 30 \quad (\text{thousand people/year})$$

(c) When is the Population Growth Rate Zero?

Set $P'(t) = 0$:

$$40 - t = 0 \Rightarrow t = 40$$

$$t = 40$$

(d) Interpretation:

Since $t = 40$ corresponds to the year $2000 + 40 = 2040$, the population stops growing in 2040.

Conclusion: In 2040, the population reaches its maximum and stops increasing. After that, it begins to decline.

$$\text{The population stops growing in 2040.}$$

Question 6

The temperature T (in $^{\circ}\text{C}$) in a chemical reaction chamber is modeled by the function

$$T(t) = 100 - 20e^{-0.5t}$$

where t is the time in minutes since the reaction started.

- (a) Find the rate at which the temperature is changing with respect to time.
- (b) What is the rate of temperature change at $t = 0$?
- (c) What is the long-term behavior of the rate of temperature change? Explain.

Question 6 - Solution

We are given:

$$T(t) = 100 - 20e^{-0.5t}$$

(a) Rate of Change of Temperature:

Differentiate $T(t)$ with respect to t :

$$T'(t) = \frac{d}{dt}(100 - 20e^{-0.5t}) = -20 \cdot \frac{d}{dt}(e^{-0.5t}) = -20 \cdot (-0.5e^{-0.5t}) = 10e^{-0.5t}$$

$$T'(t) = 10e^{-0.5t}$$

This represents the rate of temperature change at time t .

(b) Rate of Change at $t = 0$:

$$T'(0) = 10e^{-0.5 \cdot 0} = 10e^0 = 10$$

$$T'(0) = 10 \quad (\text{°C per minute})$$

Interpretation: At the beginning of the reaction, the temperature is increasing at 10°C per minute.

(c) Long-Term Behavior:

As $t \rightarrow \infty$, $e^{-0.5t} \rightarrow 0$, so:

$$T'(t) = 10e^{-0.5t} \rightarrow 0$$

Conclusion: The rate of temperature change decreases over time and approaches zero. This means the temperature stabilizes and stops changing as time progresses.

$$\lim_{t \rightarrow \infty} T'(t) = 0$$

Question 7

The position of a particle moving along a straight line is given by the function

$$s(t) = t^3 - 6t^2 + 9t$$

where $s(t)$ is in meters and $t \geq 0$ is in seconds.

- (a) Find the velocity function $v(t)$.
- (b) At what time(s) is the particle at rest?
- (c) Determine the direction of motion (moving forward or backward) on the intervals determined by part (b).
- (d) Is the particle speeding up or slowing down at $t = 1$? Justify using acceleration.

Question 7 - Solution

Velocity and rest times.

$$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3), \quad a(t) = 6t - 12.$$

Hence the particle is at rest at $t = 1, 3$.

For $t \geq 0$, it moves forward on $[0, 1) \cup (3, \infty)$ and backward on $(1, 3)$.

Behavior at $t = 1$. Here $v(1) = 0$ and $a(1) = -6$. Just before 1, velocity and acceleration have opposite signs, so speed decreases. Just after 1, both are negative, so speed increases.

The speed $|v(t)|$ has left derivative -6 and right derivative 6 at 1; it is not differentiable there.

At $t = 1$ it is at rest and reverses direction.

It slows down immediately before that instant and speeds up immediately afterward.

Question 8

The height of a ball thrown vertically upward is modeled by the function:

$$h(t) = -16t^2 + 64t + 80$$

where $h(t)$ is the height in feet after t seconds.

- (a) Determine the velocity function of the ball.
- (b) When does the ball reach its maximum height?
- (c) What is the maximum height the ball reaches?
- (d) What is the acceleration of the ball at any time t ?

Question 8 - Solution

We are given:

$$h(t) = -16t^2 + 64t + 80$$

(a) Velocity Function:

The velocity is the first derivative of height:

$$v(t) = h'(t) = -32t + 64$$

$$\boxed{v(t) = -32t + 64}$$

(b) Time of Maximum Height:

The ball reaches its maximum height when the velocity is zero:

$$-32t + 64 = 0 \Rightarrow t = 2$$

$$\boxed{t = 2 \text{ seconds}}$$

(c) Maximum Height:

Plug $t = 2$ into the original height function:

$$h(2) = -16(2)^2 + 64(2) + 80 = -64 + 128 + 80 = 144$$

$$\boxed{144 \text{ feet}}$$

(d) Acceleration Function:

Acceleration is the derivative of velocity:

$$a(t) = v'(t) = -32$$

$$\boxed{a(t) = -32 \text{ ft/s}^2 \text{ (constant acceleration due to gravity)}}$$

Question 9

The position $s(t)$ of a particle moving along a straight line is given by:

$$s(t) = \frac{3t^2 + 2}{t + 1}, \quad t > 0$$

- (a) Find the instantaneous velocity of the particle at $t = 1$.
- (b) Interpret the sign of the velocity at $t = 1$.
- (c) Find all critical points of $s(t)$ and determine whether the particle changes direction.

Question 9 - Solution

We are given:

$$s(t) = \frac{3t^2 + 2}{t + 1}, \quad t > 0$$

(a) Instantaneous Velocity at $t = 1$:

Velocity is the derivative $s'(t)$. Using the quotient rule:

$$\begin{aligned} s'(t) &= \frac{(t+1)(6t) - (3t^2+2)(1)}{(t+1)^2} \\ &= \frac{6t^2 + 6t - 3t^2 - 2}{(t+1)^2} = \frac{3t^2 + 6t - 2}{(t+1)^2} \end{aligned}$$

Evaluate at $t = 1$:

$$s'(1) = \frac{3 + 6 - 2}{4} = \frac{7}{4}$$

$$\boxed{s'(1) = \frac{7}{4}}$$

(b) Interpretation of the Sign of Velocity:

Since $s'(1) = \frac{7}{4} > 0$, the particle is moving in the positive direction at $t = 1$.

The particle is moving to the right (increasing position) at $t = 1$

(c) Critical Points and Direction Change:

Critical points occur where $s'(t) = 0$ or where $s'(t)$ is undefined.

The derivative is undefined at $t = -1$, which is outside the domain $t > 0$ and therefore ignored.

Set the numerator equal to zero:

$$3t^2 + 6t - 2 = 0$$

$$t = \frac{-6 \pm \sqrt{36 + 24}}{6} = \frac{-6 \pm \sqrt{60}}{6} = \frac{-3 \pm \sqrt{15}}{3}$$

This gives two solutions:

$$t = \frac{-3 - \sqrt{15}}{3} \quad \text{and} \quad t = \frac{-3 + \sqrt{15}}{3}$$

Since time must satisfy $t > 0$, the negative solution is rejected. The only valid critical time is:

$$\boxed{t = \frac{-3 + \sqrt{15}}{3}}$$

To determine whether the particle changes direction, test the sign of $s'(t)$ on either side of this value (within $t > 0$). The derivative changes from negative to positive, indicating a direction change.

The particle changes direction at $t = \frac{-3 + \sqrt{15}}{3}$

Question 10

Suppose that the function $C(p)$ models the total cost (in dollars) of producing p items, and is given by:

$$C(p) = 200 + 8p + 0.05p^2$$

- (a) Find and interpret $C'(p)$.
- (b) Find $C'(50)$ and interpret its meaning in context.
- (c) Determine whether the marginal cost is increasing or decreasing as p increases.

Question 10 - Solution

We are given:

$$C(p) = 200 + 8p + 0.05p^2$$

(a) Find and interpret $C'(p)$:

Differentiate:

$$C'(p) = \frac{d}{dp}(200 + 8p + 0.05p^2) = 0 + 8 + 0.1p = 8 + 0.1p$$

Interpretation: $C'(p)$ is the marginal cost — the approximate cost of producing one additional item when p items have already been produced.

(b) Compute and interpret $C'(50)$:

$$C'(50) = 8 + 0.1(50) = 8 + 5 = \boxed{13}$$

Interpretation: When 50 items have been produced, the cost of producing one more item is approximately \$13.

(c) Is marginal cost increasing or decreasing?

Since:

$$C'(p) = 8 + 0.1p$$

and the derivative of this is:

$$C''(p) = 0.1 > 0$$

The marginal cost is increasing as p increases.

Marginal cost increases with production level.