

Question 1

Use logarithmic differentiation to find the derivative of:

$$f(x) = (x^2 + 1)^x$$

Simplify your answer as much as possible.

Question 1 - Solution

We are given:

$$f(x) = (x^2 + 1)^x$$

Take the natural log of both sides:

$$\ln f(x) = \ln((x^2 + 1)^x) = x \ln(x^2 + 1)$$

Differentiate both sides implicitly:

$$\frac{1}{f(x)} \cdot f'(x) = \ln(x^2 + 1) + x \cdot \frac{1}{x^2 + 1} \cdot 2x = \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}$$

Multiply both sides by $f(x) = (x^2 + 1)^x$:

$$f'(x) = (x^2 + 1)^x \left[\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right]$$

Final Answer:

$$f'(x) = \boxed{(x^2 + 1)^x \left[\ln(x^2 + 1) + \frac{2x^2}{x^2 + 1} \right]}$$

Question 2

Let the function be:

$$f(x) = \frac{(x^2 + 1)^5 \cdot \sqrt{x + 3}}{(x - 1)^4}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your final expression as much as possible.

Question 2 - Solution

The real domain is $x \geq -3$, $x \neq 1$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |f(x)| = 5 \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - 4 \ln |x - 1|$$

Differentiating gives

$$f'(x) = f(x) \left(\frac{10x}{x^2 + 1} + \frac{1}{2(x + 3)} - \frac{4}{x - 1} \right).$$

This formula holds for $x > -3$, $x \neq 1$. At $x = -3$ there is no finite derivative because of the square-root factor.

Question 3

Let the function be:

$$f(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Clearly indicate how you apply the logarithmic properties before differentiating.

Question 3 - Solution

We are given:

$$f(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}}$$

Step 1: Take the natural log of both sides:

$$\ln f(x) = \ln \left(\frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}} \right)$$

Apply logarithmic identities:

$$\ln f(x) = 4 \ln(3x+1) + \ln(\ln x) - \frac{3}{2} \ln(x^2-9)$$

Step 2: Differentiate both sides implicitly:

$$\frac{f'(x)}{f(x)} = \frac{4 \cdot 3}{3x+1} + \frac{1}{\ln x} \cdot \frac{1}{x} - \frac{3}{2} \cdot \frac{2x}{x^2-9}$$

Simplify:

$$\frac{f'(x)}{f(x)} = \frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9}$$

Step 3: Multiply both sides by $f(x)$ to isolate $f'(x)$:

$$f'(x) = f(x) \left(\frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9} \right)$$

Substitute back the original $f(x)$:

$$f'(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}} \left(\frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9} \right)$$

$$\boxed{f'(x) = \frac{(3x+1)^4 \cdot \ln(x)}{(x^2-9)^{3/2}} \left(\frac{12}{3x+1} + \frac{1}{x \ln x} - \frac{3x}{x^2-9} \right)}$$

Question 4

For $x > 1$ (so the base is positive), let the function be:

$$f(x) = \left(\frac{x^2 \cdot \sqrt{x+1}}{(x^2+4)^3 \cdot \ln x} \right)^{\sin x}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 4 - Solution

For $x > 1$, define the positive base

$$B(x) = \frac{x^2\sqrt{x+1}}{(x^2+4)^3 \ln x}, \quad f(x) = B(x)^{\sin x}.$$

Set

$$A(x) = \ln B(x) = 2 \ln x + \frac{1}{2} \ln(x+1) - 3 \ln(x^2+4) - \ln(\ln x).$$

Then $\ln f(x) = \sin x A(x)$ and

$$A'(x) = \frac{2}{x} + \frac{1}{2(x+1)} - \frac{6x}{x^2+4} - \frac{1}{x \ln x}.$$

Logarithmic differentiation yields

$$\boxed{f'(x) = B(x)^{\sin x} (\cos x A(x) + \sin x A'(x)), \quad x > 1.}$$

Here B, A, A' are the explicit expressions above.

Question 5

Let the function be:

$$f(x) = \frac{(x^2 + 1)^5 \cdot \sqrt[3]{\ln(x)}}{e^{x^3} \cdot (x - 4)^2}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your answer as much as possible.

Question 5 - Solution

The real domain is $x > 0$, $x \neq 4$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |f(x)| = 5 \ln(x^2 + 1) + \frac{1}{3} \ln |\ln x| - x^3 - 2 \ln |x - 4|$$

Differentiating gives

$$f'(x) = f(x) \left(\frac{10x}{x^2 + 1} + \frac{1}{3x \ln x} - 3x^2 - \frac{2}{x - 4} \right).$$

This formula holds for $x > 0$, $x \neq 1, 4$. At $x = 1$, $f(1) = 0$, but the cube-root factor gives no finite derivative.

Question 6

Let the function be:

$$f(x) = \frac{(x^2 + 3x + 5)^4 \cdot \sqrt{x^2 - 1}}{(x + 2)^5 \cdot (3x - 1)^{1/3}}$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify the result as much as possible.

Question 6 - Solution

The real domain is $|x| \geq 1$, $x \neq -2$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |f(x)| = 4 \ln(x^2 + 3x + 5) + \frac{1}{2} \ln(x^2 - 1) - 5 \ln |x + 2| - \frac{1}{3} \ln |3x - 1|$$

Differentiating gives

$$f'(x) = f(x) \left(\frac{4(2x+3)}{x^2+3x+5} + \frac{x}{x^2-1} - \frac{5}{x+2} - \frac{1}{3x-1} \right).$$

This formula holds for $|x| > 1$, $x \neq -2$. At $x = \pm 1$ there is no finite derivative. The cube root is the real cube root.

Question 7

Let the function be:

$$y = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify the result as much as possible.

Question 7 - Solution

We are given:

$$y = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2$$

Take the natural logarithm of both sides:

$$\ln y = \ln \left[\left(\frac{(2x^2 + 1)^5}{(5x - 3)^{1/2} \cdot (x^2 + 4)^3} \right)^2 \right]$$

Bring the exponent outside:

$$\ln y = 2 \cdot \ln \left(\frac{(2x^2 + 1)^5}{(5x - 3)^{1/2} \cdot (x^2 + 4)^3} \right)$$

Use log rules:

$$\ln y = 2 \left[5 \ln(2x^2 + 1) - \frac{1}{2} \ln(5x - 3) - 3 \ln(x^2 + 4) \right]$$

Differentiate both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \left[\frac{5 \cdot 4x}{2x^2 + 1} - \frac{1}{2} \cdot \frac{5}{5x - 3} - \frac{3 \cdot 2x}{x^2 + 4} \right]$$

Simplify:

$$\frac{dy}{dx} = y \cdot \left(\frac{40x}{2x^2 + 1} - \frac{5}{5x - 3} - \frac{12x}{x^2 + 4} \right)$$

Substitute y back:

$$\frac{dy}{dx} = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2 \cdot \left(\frac{40x}{2x^2 + 1} - \frac{5}{5x - 3} - \frac{12x}{x^2 + 4} \right)$$

Final Answer:

$$\boxed{\frac{dy}{dx} = \left(\frac{(2x^2 + 1)^5}{\sqrt{5x - 3} \cdot (x^2 + 4)^3} \right)^2 \cdot \left(\frac{40x}{2x^2 + 1} - \frac{5}{5x - 3} - \frac{12x}{x^2 + 4} \right)}$$

Question 8

Let the function be:

$$y = (x^2 + 3x + 1)^4 \cdot \sqrt{x^2 - 1} \cdot \frac{1}{(2x + 5)^3}$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify your answer as much as possible.

Question 8 - Solution

Put $A = x^2 + 3x + 1$. The real domain is $|x| \geq 1$, $x \neq -5/2$.

For $A \neq 0$ and $|x| > 1$, logarithmic differentiation gives

$$\ln |y| = 4 \ln |A| + \frac{1}{2} \ln(x^2 - 1) - 3 \ln |2x + 5|,$$

$$\frac{y'}{y} = \frac{4(2x + 3)}{A} + \frac{x}{x^2 - 1} - \frac{6}{2x + 5}.$$

Cancel before evaluating at zeros of A :

$$y' = \frac{A^3 [4(2x + 3)(x^2 - 1)(2x + 5) + Ax(2x + 5) - 6A(x^2 - 1)]}{\sqrt{x^2 - 1}(2x + 5)^4}.$$

This formula holds for $|x| > 1$, $x \neq -5/2$, including $x = (-3 - \sqrt{5})/2$, where $y' = 0$. The other root of A is outside the domain.

There is no finite derivative at $x = \pm 1$.

Question 9

Let

$$y = \frac{\sqrt{(x^2 + 4)^3(x - 1)^2}}{(x^2 - 1)^4}$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify your final answer fully.

Question 9 - Solution

For $x \neq \pm 1$, the square root simplifies to

$$y = \frac{(x^2 + 4)^{3/2}|x - 1|}{(x^2 - 1)^4} > 0.$$

Hence

$$\ln y = \frac{3}{2} \ln(x^2 + 4) + \ln|x - 1| - 4 \ln|x^2 - 1|.$$

Differentiating and multiplying by y gives

$$y' = \frac{(x^2 + 4)^{3/2}|x - 1|}{(x^2 - 1)^4} \left(\frac{3x}{x^2 + 4} + \frac{1}{x - 1} - \frac{8x}{x^2 - 1} \right), \quad x \neq \pm 1.$$

Question 10

Let

$$y = \frac{(x^2 + 1)^5 \cdot \sqrt[3]{x^2 - 4}}{\sqrt{2x + 1} \cdot (x - 3)^2}$$

- (a) Use logarithmic differentiation to find $\frac{dy}{dx}$.
- (b) Simplify your answer fully.

Question 10 - Solution

The real domain is $x > -\frac{1}{2}$, $x \neq 3$. Where the function is nonzero, use absolute values in logarithmic differentiation:

$$\ln |y| = 5 \ln(x^2 + 1) + \frac{1}{3} \ln |x^2 - 4| - \frac{1}{2} \ln(2x + 1) - 2 \ln |x - 3|$$

Differentiating gives

$$y' = y \left(\frac{10x}{x^2 + 1} + \frac{2x}{3(x^2 - 4)} - \frac{1}{2x + 1} - \frac{2}{x - 3} \right).$$

This formula holds for $x > -\frac{1}{2}$, $x \neq 2, 3$. At $x = 2$, $y = 0$, but there is no finite derivative because of the cube-root factor.