

Question 1

Let:

$$f(x) = x^4 - 4x^3 + 6x^2$$

- (a) Find the first and second derivatives of $f(x)$.
- (b) Determine the intervals on which $f(x)$ is concave up and concave down.
- (c) Find all points of inflection.

Question 1 - Solution

We are given:

$$f(x) = x^4 - 4x^3 + 6x^2$$

(a) First derivative:

$$f'(x) = 4x^3 - 12x^2 + 12x$$

Second derivative:

$$f''(x) = 12x^2 - 24x + 12$$

Factor:

$$f''(x) = 12(x^2 - 2x + 1) = 12(x - 1)^2$$

Answer:

$$f'(x) = 4x^3 - 12x^2 + 12x, \quad f''(x) = 12(x - 1)^2$$

(b) Analyze sign of $f''(x)$:

Since $f''(x) = 12(x - 1)^2 \geq 0$ for all x , the second derivative is never negative.

$f''(x) > 0$ for $x \neq 1 \rightarrow$ concave up

$f''(1) = 0 \rightarrow$ possible inflection point

But since the sign does not change around $x = 1$, it is not an inflection point.

Answer: Concave up: $(-\infty, \infty)$ Concave down: None

(c) No change in concavity at $x = 1$, so:

Answer: No points of inflection

Question 2

Let $f(x) = x^2e^{-x}$.

- (a) Compute the first four derivatives of $f(x)$.
- (b) Identify a pattern in the derivatives.
- (c) Use the pattern to write a general formula for $f^{(n)}(x)$ for $n \geq 0$.

Question 2 - Solution

We are given $f(x) = x^2e^{-x}$. Let's compute derivatives step-by-step.

(a) First, compute the derivatives:

First derivative:

$$f'(x) = \frac{d}{dx}(x^2e^{-x}) = 2xe^{-x} + x^2(-e^{-x}) = (2x - x^2)e^{-x}$$

Second derivative:

$$f''(x) = \frac{d}{dx}[(2x - x^2)e^{-x}] = [2 - 2x]e^{-x} + (2x - x^2)(-e^{-x}) = [2 - 2x - 2x + x^2]e^{-x} = (x^2 - 4x + 2)e^{-x}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[(x^2 - 4x + 2)e^{-x}] = (2x - 4)e^{-x} + (x^2 - 4x + 2)(-e^{-x}) = (-x^2 + 6x - 6)e^{-x}$$

Fourth derivative:

$$f^{(4)}(x) = \frac{d}{dx}[(-x^2 + 6x - 6)e^{-x}] = (-2x + 6)e^{-x} + (-x^2 + 6x - 6)(-e^{-x}) = (x^2 - 8x + 12)e^{-x}$$

(b) Pattern:

Each derivative is a polynomial times e^{-x} . The polynomials alternate in sign and follow a recursive-like behavior:

$$\begin{aligned}f(x) &= x^2e^{-x} \\f'(x) &= (2x - x^2)e^{-x} \\f''(x) &= (x^2 - 4x + 2)e^{-x} \\f^{(3)}(x) &= (-x^2 + 6x - 6)e^{-x} \\f^{(4)}(x) &= (x^2 - 8x + 12)e^{-x}\end{aligned}$$

(c) General formula (structure):

Let:

$$f^{(n)}(x) = P_n(x)e^{-x}$$

Where $P_n(x)$ is a polynomial of degree 2, alternating in sign:

$$P_n(x) = (-1)^n(x^2 - 2nx + n(n-1))$$

Final boxed result:

$$\boxed{f^{(n)}(x) = (-1)^n(x^2 - 2nx + n(n-1))e^{-x}}$$

Question 3

Let $f(x) = \ln(x^2 + 1)$.

- (a) Compute the first three derivatives of $f(x)$.
- (b) Determine the value of $f^{(3)}(0)$.
- (c) Based on the derivatives at $x = 0$, write the third-degree Taylor polynomial centered at $x = 0$.

Question 3 - Solution

We are given:

$$f(x) = \ln(x^2 + 1)$$

(a) First three derivatives:

First derivative:

$$f'(x) = \frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}$$

Second derivative:

$$f''(x) = \frac{d}{dx} \left(\frac{2x}{x^2 + 1} \right) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{2(x^2 + 1 - 2x^2)}{(x^2 + 1)^2} = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Third derivative:

We apply the quotient rule to:

$$f''(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2}$$

Let's compute:

$$f^{(3)}(x) = \frac{d}{dx} \left(\frac{2(1 - x^2)}{(x^2 + 1)^2} \right)$$

Numerator derivative:

$$\frac{d}{dx} [2(1 - x^2)] = -4x$$

Denominator derivative:

$$\frac{d}{dx} [(x^2 + 1)^2] = 2(x^2 + 1)(2x) = 4x(x^2 + 1)$$

Now apply quotient rule:

$$f^{(3)}(x) = \frac{(-4x)(x^2 + 1)^2 - 2(1 - x^2)(4x)(x^2 + 1)}{(x^2 + 1)^4}$$

Factor out $4x(x^2 + 1)$:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1) [(x^2 + 1) + 2(1 - x^2)]}{(x^2 + 1)^4}$$

Simplify:

$$(x^2 + 1) + 2(1 - x^2) = x^2 + 1 + 2 - 2x^2 = 3 - x^2$$

So:

$$f^{(3)}(x) = \frac{-4x(x^2 + 1)(3 - x^2)}{(x^2 + 1)^4} = \frac{-4x(3 - x^2)}{(x^2 + 1)^3}$$

(b) Evaluate at $x = 0$:

$$f(0) = \ln(1) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f^{(3)}(0) = \frac{-4(0)(3 - 0)}{1^3} = 0$$

(c) Taylor polynomial of degree 3 at $x = 0$:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 = 0 + 0 + \frac{2}{2}x^2 + 0 = x^2$$

$$\boxed{T_3(x) = x^2}$$

Question 4

Let $f(x) = e^x \cos(x)$.

- (a) Compute the first three derivatives: $f'(x)$, $f''(x)$, and $f^{(3)}(x)$.
- (b) Find the value of $f''(0)$.
- (c) Determine the third-degree Taylor polynomial of $f(x)$ centered at $x = 0$.

Question 4 - Solution

We are given:

$$f(x) = e^x \cos(x)$$

(a) **First three derivatives:**

Use product rule: $f(x) = u(x)v(x)$, where $u = e^x$, $v = \cos(x)$

First derivative:

$$f'(x) = u'v + uv' = e^x \cos(x) - e^x \sin(x) = e^x(\cos(x) - \sin(x))$$

Second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}[e^x(\cos(x) - \sin(x))] = e^x(\cos(x) - \sin(x)) + e^x(-\sin(x) - \cos(x)) \\ &= e^x[(\cos(x) - \sin(x)) + (-\sin(x) - \cos(x))] = e^x(-2\sin(x)) \end{aligned}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}[e^x(-2\sin(x))] = e^x(-2\sin(x)) + e^x(-2\cos(x)) = -2e^x(\sin(x) + \cos(x))$$

(b) **Evaluate $f''(0)$:**

$$f''(0) = e^0(-2\sin(0)) = 1 \cdot 0 = \boxed{0}$$

(c) **Taylor polynomial of degree 3 at $x = 0$:**

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3$$

Compute values:

$$f(0) = e^0 \cos(0) = 1, \quad f'(0) = e^0(\cos(0) - \sin(0)) = 1, \quad f''(0) = 0$$

$$f^{(3)}(0) = -2e^0(\sin(0) + \cos(0)) = -2(0 + 1) = -2$$

So:

$$T_3(x) = 1 + x + 0 + \frac{-2}{6}x^3 = 1 + x - \frac{1}{3}x^3$$

$$\boxed{T_3(x) = 1 + x - \frac{1}{3}x^3}$$

Question 5

Let $f(x) = \sin(x^2)$.

- (a) Find $f'(x)$, $f''(x)$, and $f^{(3)}(x)$.
- (b) Compute $f''(0)$.
- (c) Use your results to write the second-degree Taylor polynomial of $f(x)$ centered at $x = 0$.

Question 5 - Solution

We are given:

$$f(x) = \sin(x^2)$$

(a) First three derivatives:

First derivative:

$$f'(x) = \frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2)$$

Second derivative: Use product rule:

$$f''(x) = \frac{d}{dx} (2x \cos(x^2)) = 2 \cos(x^2) + 2x \cdot (-\sin(x^2)) \cdot 2x = 2 \cos(x^2) - 4x^2 \sin(x^2)$$

Third derivative: Differentiate $f''(x)$:

$$f^{(3)}(x) = \frac{d}{dx} (2 \cos(x^2) - 4x^2 \sin(x^2))$$

First term:

$$\frac{d}{dx} [2 \cos(x^2)] = -2 \sin(x^2) \cdot 2x = -4x \sin(x^2)$$

Second term (product rule):

$$\frac{d}{dx} [4x^2 \sin(x^2)] = 8x \sin(x^2) + 4x^2 \cdot \cos(x^2) \cdot 2x = 8x \sin(x^2) + 8x^3 \cos(x^2)$$

So:

$$f^{(3)}(x) = -4x \sin(x^2) - (8x \sin(x^2) + 8x^3 \cos(x^2)) = -12x \sin(x^2) - 8x^3 \cos(x^2)$$

(b) Compute $f''(0)$:

$$f''(x) = 2 \cos(x^2) - 4x^2 \sin(x^2) \Rightarrow f''(0) = 2 \cos(0) - 0 = 2$$

(c) Taylor Polynomial $T_2(x)$:

We use:

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$f(0) = \sin(0) = 0, \quad f'(0) = 2(0) \cos(0) = 0, \quad f''(0) = 2$$

So:

$$T_2(x) = 0 + 0 + \frac{2}{2}x^2 = x^2$$

$$\boxed{T_2(x) = x^2}$$

Question 6

Let $f(x) = \frac{1}{1-x}$.

- (a) Find the first four derivatives $f'(x), f''(x), f^{(3)}(x), f^{(4)}(x)$.
- (b) Find a general formula for $f^{(n)}(x)$.
- (c) Evaluate $f^{(n)}(0)$ and write the Taylor series for $f(x)$ centered at $x = 0$.

Question 6 - Solution

We are given:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

(a) Compute derivatives:

Using the chain rule,

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{d}{dx}(1-x)^{-2} = -2(1-x)^{-3} \cdot (-1) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{d}{dx} \left(\frac{2}{(1-x)^3} \right) = 2 \cdot 3(1-x)^{-4} = \frac{6}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{d}{dx} \left(\frac{6}{(1-x)^4} \right) = 6 \cdot 4(1-x)^{-5} = \frac{24}{(1-x)^5}$$

(b) General formula:

Each derivative introduces a factorial factor and increases the power of the denominator by 1:

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

(c) Evaluate at $x = 0$:

$$f^{(n)}(0) = \frac{n!}{1^{n+1}} = n!$$

Thus the Taylor series centered at $x = 0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Question 7

Let $f(x) = x^2e^x$.

- (a) Find the first three derivatives of $f(x)$.
- (b) Find a general expression for $f^{(n)}(x)$.
- (c) Evaluate $f^{(3)}(0)$.

Question 7 - Solution

We are given:

$$f(x) = x^2 e^x$$

(a) Compute Derivatives:

We use the product rule: If $u = x^2$, $v = e^x$, then $f = uv$

$$f'(x) = (x^2)'e^x + x^2(e^x) = 2xe^x + x^2e^x = e^x(2x + x^2)$$

$$f''(x) = \frac{d}{dx}[e^x(2x + x^2)] = e^x(2 + 2x + 2x + x^2) = e^x(x^2 + 4x + 2)$$

$$f^{(3)}(x) = \frac{d}{dx}[e^x(x^2 + 4x + 2)] = e^x(x^2 + 4x + 2 + 2x + 4 + 0) = e^x(x^2 + 6x + 6)$$

(b) General Formula:

We notice a pattern in each derivative:

$$f^{(n)}(x) = e^x \cdot P_n(x)$$

where $P_n(x)$ is a degree-2 polynomial (since the original function had x^2).

This suggests:

$$\boxed{f^{(n)}(x) = e^x \cdot (x^2 + 2nx + n(n-1))}$$

(This can be proven by induction.)

(c) Evaluate $f^{(3)}(0)$:

From (a):

$$f^{(3)}(0) = e^0(0^2 + 6 \cdot 0 + 6) = 1 \cdot 6 = \boxed{6}$$

Question 8

Let $f(x) = \frac{1}{x}$.

- (a) Find the first four derivatives of $f(x)$.
- (b) Based on the pattern, write a general formula for the n -th derivative $f^{(n)}(x)$.
- (c) Evaluate $f^{(6)}(x)$.

Question 8 - Solution

We are given:

$$f(x) = \frac{1}{x} = x^{-1}$$

(a) Compute derivatives:

$$f'(x) = \frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$f''(x) = \frac{d}{dx}(-x^{-2}) = 2x^{-3}$$

$$f^{(3)}(x) = \frac{d}{dx}(2x^{-3}) = -6x^{-4}$$

$$f^{(4)}(x) = \frac{d}{dx}(-6x^{-4}) = 24x^{-5}$$

(b) General Pattern:

Observe the signs and coefficients:

$$f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-(n+1)}$$

Answer: $f^{(n)}(x) = (-1)^n \cdot n! \cdot x^{-(n+1)}$

(c) Evaluate $f^{(6)}(x)$:

Use the formula:

$$f^{(6)}(x) = (-1)^6 \cdot 6! \cdot x^{-7} = 720x^{-7}$$

Answer: $f^{(6)}(x) = \frac{720}{x^7}$

Question 9

Let $f(x) = \cos(2x)$.

- (a) Find the first four derivatives of $f(x)$.
- (b) Find a general formula for the n -th derivative $f^{(n)}(x)$.
- (c) Evaluate $f^{(15)}(x)$.

Question 9 - Solution

Successive differentiation yields

$$f' = -2 \sin 2x, \quad f'' = -4 \cos 2x, \quad f^{(3)} = 8 \sin 2x, \quad f^{(4)} = 16 \cos 2x.$$

Each derivative multiplies by 2 and advances the phase by $\pi/2$; induction therefore gives

$$\boxed{f^{(n)}(x) = 2^n \cos(2x + n\pi/2)}.$$

Since $15\pi/2$ is congruent to $3\pi/2$ modulo 2π ,

$$\boxed{f^{(15)}(x) = 32768 \cos(2x + 3\pi/2) = 32768 \sin 2x}.$$

Question 10

Let

$$f(x) = \ln(1 + x^2).$$

- (a) Show that all odd-order derivatives of $f(x)$ evaluated at $x = 0$ are zero.
- (b) Find a closed-form expression for the $2n$ -th derivative of $f(x)$ evaluated at $x = 0$, where $n \in \mathbb{N}$.
- (c) Using your result from part (b), write the Maclaurin series for $f(x)$ in summation notation.
- (d) Determine the radius of convergence of the Maclaurin series and justify your answer.
- (e) Use the Maclaurin series to approximate $\ln(1.1)$ to within an error of 10^{-4} , and explain how you control the error.

Question 10 - Solution

We are given

$$f(x) = \ln(1 + x^2).$$

(a) Odd-order derivatives at $x = 0$

Observe that $f(x)$ is an **even function**, since

$$f(-x) = \ln(1 + (-x)^2) = \ln(1 + x^2) = f(x).$$

The derivative of an even function is odd, and the derivative of an odd function is even. Therefore:

- $f'(x)$ is odd,
- $f''(x)$ is even,
- $f^{(3)}(x)$ is odd,
- and so on.

All odd functions evaluate to zero at $x = 0$. Hence,

$$f^{(2k+1)}(0) = 0 \quad \text{for all } k \geq 0.$$

(b) Closed form for $f^{(2n)}(0)$

We begin with the known Taylor expansion:

$$\ln(1 + u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n}, \quad |u| < 1.$$

Substitute $u = x^2$:

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}.$$

Comparing this with the Maclaurin series definition:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

we see that only even powers appear. Matching coefficients of x^{2n} , we obtain:

$$\frac{f^{(2n)}(0)}{(2n)!} = (-1)^{n+1} \frac{1}{n}.$$

Solving for $f^{(2n)}(0)$:

$$f^{(2n)}(0) = (-1)^{n+1} \frac{(2n)!}{n}.$$

(c) Maclaurin series

Using the result above, the Maclaurin series for $f(x)$ is:

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}, \quad |x| < 1.$$

(d) Radius of convergence

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$$

is a power series in x^2 .

Using the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2(n+1)}/(n+1)}{x^{2n}/n} \right| = |x|^2.$$

The series converges when:

$$|x|^2 < 1 \quad \Rightarrow \quad |x| < 1.$$

Thus, the radius of convergence is:

$$\boxed{R = 1.}$$

(e) Approximating $\ln(1.1)$

We write:

$$\ln(1.1) = \ln(1 + 0.1) = \ln(1 + x^2) \quad \text{with } x^2 = 0.1.$$

Using the alternating series:

$$\ln(1.1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}.$$

Since this is an alternating series with decreasing terms, the error after N terms is bounded by the magnitude of the next term:

$$|R_N| \leq \frac{(0.1)^{N+1}}{N+1}.$$

We test values:

$$\frac{(0.1)^3}{3} \approx 0.00033 > 10^{-4}, \quad \frac{(0.1)^4}{4} = 0.000025 < 10^{-4}.$$

Thus, using the first four terms guarantees the desired accuracy.

$$\boxed{\ln(1.1) \approx 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4}}$$

