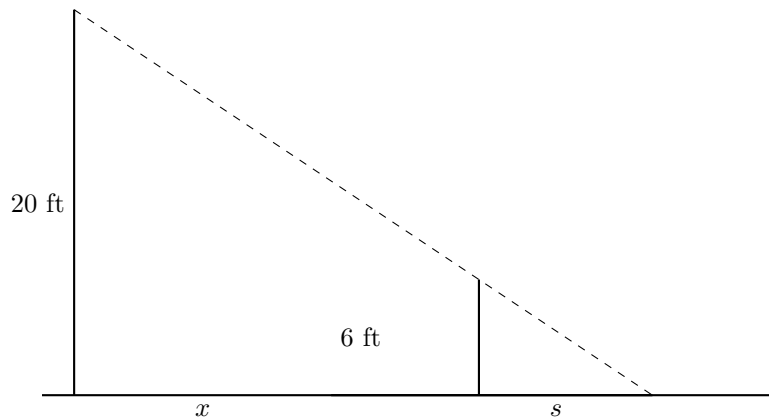


Question 1

A streetlight is mounted at the top of a 20-foot pole. A person 6 feet tall walks directly away from the pole at a constant rate of 4 ft/sec.

- (a) How fast is the tip of the person's shadow moving along the ground when the person is 10 feet from the pole?
- (b) How fast is the length of the person's shadow changing at that instant?
- (c) Explain why the tip of the shadow moves faster than the person.



Question 1 - Solution

Let:

$x(t)$ = distance from the pole to the person

$s(t)$ = length of the shadow

The distance from the pole to the tip of the shadow is:

$$x + s$$

Step 1: Set up the relationship

Using similar triangles:

$$\frac{20}{x + s} = \frac{6}{s}$$

Cross-multiply:

$$20s = 6(x + s)$$

Simplify:

$$20s = 6x + 6s \Rightarrow 14s = 6x \Rightarrow s = \frac{3}{7}x$$

Step 2: Differentiate with respect to time

Differentiate both sides:

$$\frac{ds}{dt} = \frac{3}{7} \frac{dx}{dt}$$

Given:

$$\frac{dx}{dt} = 4$$

So:

$$\frac{ds}{dt} = \frac{12}{7}$$

(a) Speed of the tip of the shadow

$$\frac{d}{dt}(x + s) = \frac{dx}{dt} + \frac{ds}{dt} = 4 + \frac{12}{7} = \frac{40}{7}$$

Answer (a):

$$\boxed{\frac{40}{7} \text{ ft/sec}}$$

(b) Rate of change of the shadow length

$$\boxed{\frac{ds}{dt} = \frac{12}{7} \text{ ft/sec}}$$

(c) Explanation

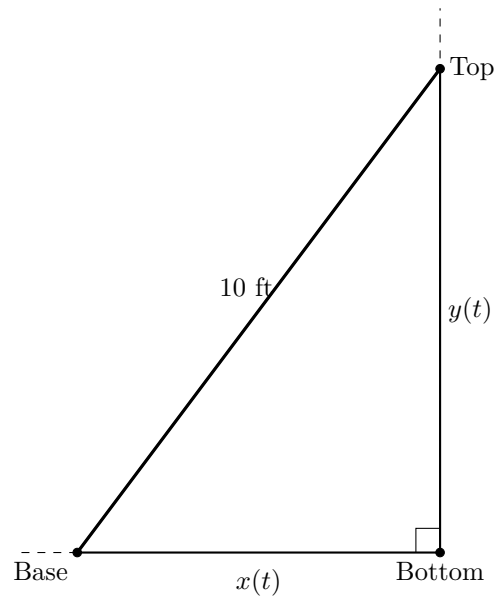
The shadow tip moves faster than the person because its motion includes both the person's movement and the changing length of the shadow.

Conclusion: The shadow tip moves faster than the person.

Question 2

A ladder 10 feet long is leaning against a vertical wall. The bottom of the ladder is sliding away from the wall at a rate of 2 ft/sec.

- (a) How fast is the top of the ladder sliding down the wall when the bottom is 6 feet from the wall?



Question 2 - Solution

Let: $x(t)$: distance from the wall to the bottom of the ladder (horizontal leg), $y(t)$: height of the ladder on the wall (vertical leg), Ladder: 10 ft, so by Pythagoras:

$$x^2 + y^2 = 10^2 = 100$$

Differentiate both sides w.r.t. time t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

Given: $x = 6$, Ladder is 10 ft, so $y = \sqrt{100 - 36} = 8$, $\frac{dx}{dt} = 2$

$$\frac{dy}{dt} = -\frac{6}{8} \cdot 2 = -\frac{12}{8} = -\frac{3}{2}$$

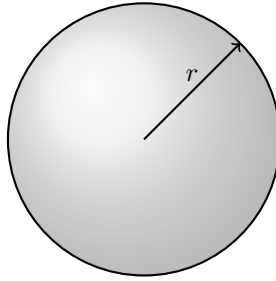
Answer:

$\frac{dy}{dt} = -\frac{3}{2} \text{ ft/sec}$

Question 3**Problem:**

A spherical balloon is being inflated such that its radius increases at a constant rate of $\frac{dr}{dt} = 3$ cm/min.

- (a) At what rate is the volume of the balloon increasing when the radius is 10 cm?
- (b) Include units and interpret the meaning of your result.



Spherical balloon

Question 3 - Solution

We are given:

$$\frac{dr}{dt} = 3 \text{ cm/min}, \quad r = 10 \text{ cm}$$

The volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3$$

Differentiate both sides with respect to t :

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3}\pi r^3 \right) = 4\pi r^2 \frac{dr}{dt}$$

Substitute the known values:

$$\frac{dV}{dt} = 4\pi(10)^2(3) = 4\pi(100)(3) = 1200\pi$$

$$\frac{dV}{dt} \approx 3769.91 \text{ cm}^3/\text{min}$$

Answer: The volume is increasing at a rate of $\boxed{1200\pi \approx 3769.91 \text{ cm}^3/\text{min}}$ when the radius is 10 cm.

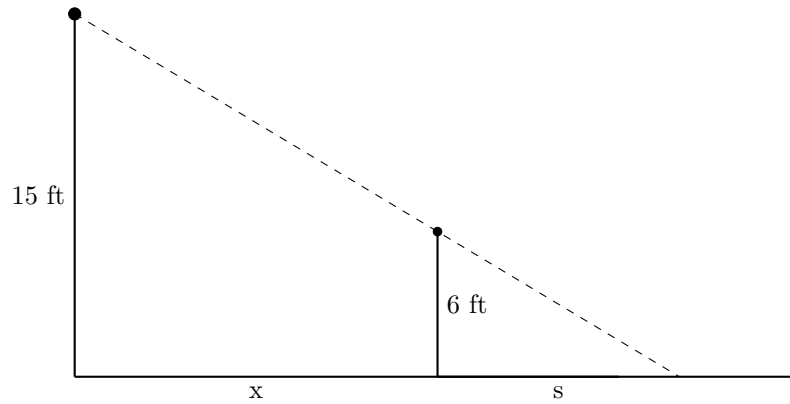
Interpretation: When the balloon's radius is 10 cm and growing at 3 cm per minute, the volume is expanding at approximately 3769.91 cubic centimeters per minute. This tells us how rapidly the space inside the balloon is increasing at that instant.

Question 4

Problem:

A 6-ft tall person walks away from a 15-ft tall streetlight at a rate of 3 ft/s.

- (a) How fast is the tip of the shadow moving when the person is 30 ft from the base of the light?
(b) At the same moment, how fast is the length of the shadow increasing?



Question 4 - Solution

Let: - $x(t)$ be the distance from the person to the light pole - $s(t)$ be the length of the shadow - The tip of the shadow is at a distance $x + s$ from the pole

Using similar triangles:

$$\frac{6}{s} = \frac{15}{x+s} \Rightarrow 6(x+s) = 15s \Rightarrow 6x + 6s = 15s \Rightarrow 6x = 9s \Rightarrow s = \frac{2}{3}x$$

Differentiate with respect to t :

$$\frac{ds}{dt} = \frac{2}{3} \frac{dx}{dt}$$

Given:

$$\frac{dx}{dt} = 3 \text{ ft/s} \Rightarrow \frac{ds}{dt} = \frac{2}{3}(3) = 2 \text{ ft/s}$$

Now, the tip of the shadow moves at:

$$\frac{d}{dt}(x+s) = \frac{dx}{dt} + \frac{ds}{dt} = 3 + 2 = \boxed{5 \text{ ft/s}}$$

Answers:

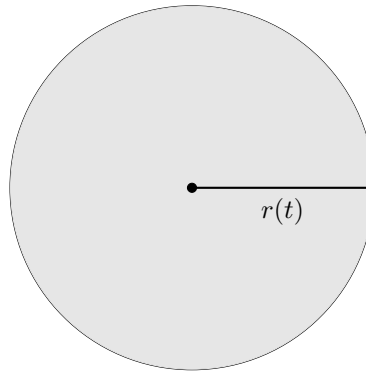
(a) The tip of the shadow is moving at $\boxed{5 \text{ ft/s}}$

(b) The shadow is lengthening at $\boxed{2 \text{ ft/s}}$

Question 5

A circular ripple is formed on the surface of a still pond when a pebble is dropped into it. The radius of the ripple increases at a constant rate of 0.05 m/s.

- (a) At what rate is the area enclosed by the ripple increasing when the radius is 0.3 m?
- (b) At what rate is the circumference of the ripple increasing at that moment?



Expanding ripple with changing radius

Question 5 - Solution

Let $r(t)$ be the radius of the ripple at time t , in meters.

(a) Rate of Change of Area

The area of a circle is:

$$A = \pi r^2$$

Differentiate both sides with respect to time t :

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We're given:

$$\frac{dr}{dt} = 0.05 \text{ m/s}, \quad r = 0.3 \text{ m}$$

Substitute:

$$\frac{dA}{dt} = 2\pi(0.3)(0.05) = 0.03\pi \approx \boxed{0.0942 \text{ m}^2/\text{s}}$$

(b) Rate of Change of Circumference

Circumference is given by:

$$C = 2\pi r$$

Differentiate:

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

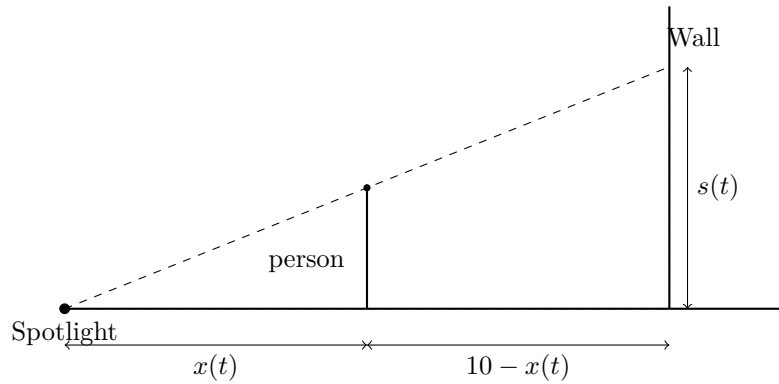
Substitute:

$$\frac{dC}{dt} = 2\pi(0.05) = \boxed{0.1\pi \approx 0.3142 \text{ m/s}}$$

Question 6

A spotlight is located on the ground 10 m away from a vertical wall. A person 2 m tall walks directly away from the wall at a speed of 1.2 m/s.

- (a) How fast is the length of the shadow on the wall changing when the person is 5 m from the wall?
- (b) At that moment, how fast is the tip of the shadow moving up the wall?



Question 6 - Solution

Let:

$x(t)$ = horizontal distance from the spotlight to the person (m)

$s(t)$ = height of the shadow on the wall (m)

From similar triangles:

$$\frac{s(t)}{10} = \frac{2}{x(t)} \Rightarrow s(t) = \frac{20}{x(t)}$$

Differentiate both sides:

$$\frac{ds}{dt} = -\frac{20}{x^2} \cdot \frac{dx}{dt}$$

Given:

$$x = 5 \text{ m}, \quad \frac{dx}{dt} = -1.2 \text{ m/s}$$

$$\frac{ds}{dt} = -\frac{20}{25}(-1.2) = 0.96 \text{ m/s}$$

(a) Rate of change of shadow height:

$\frac{ds}{dt} = 0.96 \text{ m/s}$

(b) Speed of the tip of the shadow:

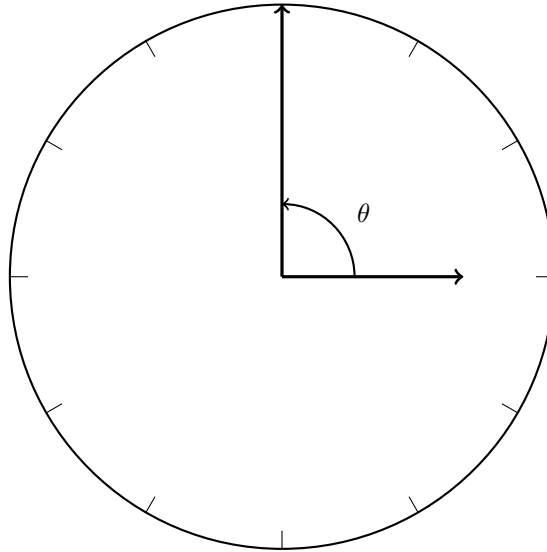
The tip moves upward at the same rate:

0.96 m/s upward

Question 7

The minute hand of a clock completes one full revolution every 60 minutes, and the hour hand completes one full revolution every 12 hours.

Question: At exactly 3:00, how fast is the *smaller angle* between the hour and minute hands changing?



Solution — Method 1 (Formal Related Rates)

Let $\theta(t)$ be the signed angle (in degrees) from the hour hand to the minute hand.

$$\omega_m = 360/60 = 6^\circ/\text{min}, \quad \omega_h = 360/(12 * 60) = 0.5^\circ/\text{min}$$

At 3:00, the hour hand is at 90° and the minute hand is at 0° .

$$\theta(t) = (6t) - (90 + 0.5t)$$

Differentiate:

$$\frac{d\theta}{dt} = 6 - 0.5 = 5.5$$

Since the smaller angle is initially decreasing,

$$\boxed{\frac{d\theta}{dt} = -5.5^\circ/\text{min}}$$

Question 7 - Solution)

The angle between the hands changes at the difference of their angular speeds.

$$\text{Minute hand speed} = 6^\circ/\text{min}$$

$$\text{Hour hand speed} = 0.5^\circ/\text{min}$$

So the angle closes at:

$$6 - 0.5 = 5.5^\circ/\text{min}$$

Since the hands are moving toward each other at 3:00, the angle is decreasing.

$$\boxed{5.5^\circ/\text{min (decreasing)}}$$

Final Answer

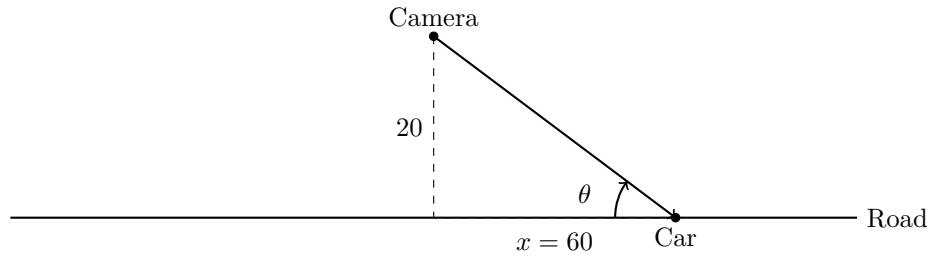
$$\boxed{\frac{d\theta}{dt} = -5.5^\circ/\text{min}}$$

The negative sign indicates the angle between the hands is decreasing.

Question 8

A camera is placed 20 meters from a straight road. A car is driving along the road away from the point closest to the camera at a speed of 30 m/s.

How fast is the angle between the camera's line of sight and the road changing when the car is 60 meters from the point on the road closest to the camera?



Question 8 - Solution

Let:

$x(t)$: horizontal distance from car to closest point on road to camera , $\theta(t)$: angle between the camera's line of sight and the road , The camera is 20 m from the road (fixed)

We have:

$$\tan(\theta) = \frac{20}{x}$$

Differentiate both sides with respect to time:

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{-20}{x^2} \cdot \frac{dx}{dt}$$

At the instant:

$x = 60$, $\frac{dx}{dt} = 30$ m/s (moving away) , $\tan(\theta) = \frac{20}{60} = \frac{1}{3} \Rightarrow \theta \approx 18.43^\circ$

Use identity:

$$\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$$

Now plug in:

$$\frac{10}{9} \cdot \frac{d\theta}{dt} = \frac{-20}{60^2} \cdot 30 = \frac{-20 \cdot 30}{3600} = \frac{-600}{3600} = -\frac{1}{6}$$

So:

$$\frac{d\theta}{dt} = \boxed{-\frac{9}{60} = -0.15 \text{ radians per second}}$$

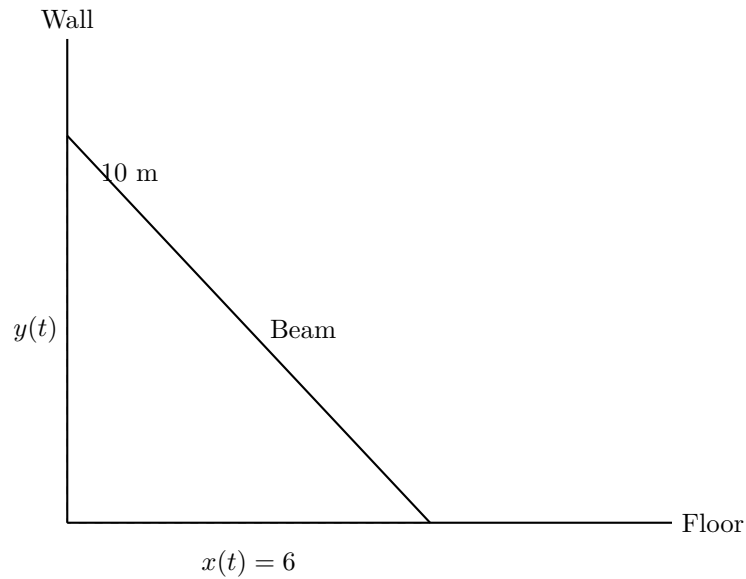
Conclusion: The angle between the camera's line of sight and the road is decreasing at a rate of:

$$\boxed{0.15 \text{ radians/second}}$$

Question 9

A 10-meter-long beam is leaning against a vertical wall and a horizontal floor, forming a right triangle. The bottom of the beam is sliding away from the wall along the floor at a rate of 1.5 m/s.

How fast is the top of the beam sliding down the wall when the bottom is 6 meters away from the wall?



Question 9 - Solution

Let:

$x(t)$: distance from the base of the wall to the bottom of the beam

$y(t)$: height of the beam on the wall

The beam is 10 m long $\Rightarrow x^2 + y^2 = 100$

Differentiate both sides with respect to time:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Given: $x = 6$, $\frac{dx}{dt} = 1.5$

Find y from Pythagoras:

$$y = \sqrt{100 - 6^2} = \sqrt{64} = 8$$

Now solve for $\frac{dy}{dt}$:

$$6(1.5) + 8 \frac{dy}{dt} = 0 \Rightarrow 9 + 8 \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{9}{8}$$

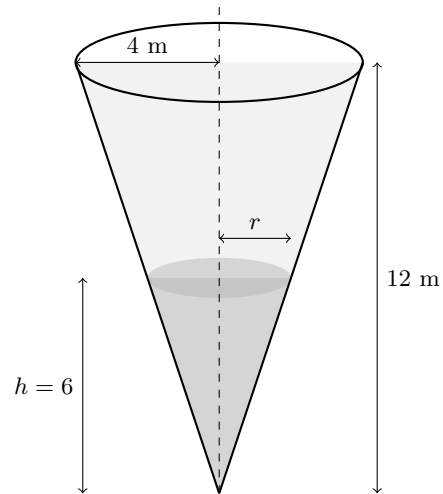
Conclusion: The top of the beam is sliding down at a rate of:

$\frac{dy}{dt} = -\frac{9}{8} = -1.125 \text{ m/s}$

Question 10

Water is poured into an inverted conical tank at a rate of $3 \text{ m}^3/\text{min}$. The tank is 12 meters deep and has a top radius of 4 meters.

How fast is the water level rising when the water is 6 meters deep?



Question 10 - Solution

Let:

V : volume of water

r : radius of water surface

h : depth of water

Given:

$$\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$$

Tank dimensions: height = 12 m, top radius = 4 m

$$\text{Volume of cone: } V = \frac{1}{3}\pi r^2 h$$

Use similar triangles:

$$\frac{r}{h} = \frac{4}{12} = \frac{1}{3} \Rightarrow r = \frac{h}{3}$$

Substitute into volume formula:

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3$$

Differentiate:

$$\frac{dV}{dt} = \frac{\pi}{9}h^2 \cdot \frac{dh}{dt}$$

Substitute known values:

$$3 = \frac{\pi}{9}(6)^2 \cdot \frac{dh}{dt} = \frac{\pi}{9} \cdot 36 \cdot \frac{dh}{dt} \Rightarrow 3 = 4\pi \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{3}{4\pi}$$

Final Answer:

$$\boxed{\frac{dh}{dt} = \frac{3}{4\pi} \text{ m/min} \approx 0.239 \text{ m/min}}$$