

Question 1

Given the equation:

$$x^2 + xy + y^2 = 7$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line at the point $(1, 2)$.

Question 1 - Solution

We are given:

$$x^2 + xy + y^2 = 7$$

Differentiate both sides with respect to x , using the product rule on xy :

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$

$$2x + \left(x \frac{dy}{dx} + y\right) + 2y \frac{dy}{dx} = 0$$

Group $\frac{dy}{dx}$ terms:

$$x \frac{dy}{dx} + 2y \frac{dy}{dx} = -2x - y \Rightarrow (x + 2y) \frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \boxed{\frac{-2x - y}{x + 2y}}$$

(b) Evaluate at the point $(1, 2)$:

$$\frac{dy}{dx} = \frac{-2(1) - 2}{1 + 2(2)} = \frac{-4}{5}$$

Answer: Slope of the tangent line at $(1, 2)$ is $\boxed{-\frac{4}{5}}$

Question 2

Let the equation of a curve be defined implicitly by:

$$x^3 + y^3 = 6xy$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line at the point $(3, 3)$.

Question 2 - Solution

We are given:

$$x^3 + y^3 = 6xy$$

(a) Differentiate both sides with respect to x :

Left side:

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3x^2 + 3y^2 \frac{dy}{dx}$$

Right side (Product Rule):

$$\frac{d}{dx}(6xy) = 6x \frac{dy}{dx} + 6y$$

Now equate both sides:

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x \frac{dy}{dx} + 6y$$

Bring like terms together:

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

Factor out $\frac{dy}{dx}$:

$$(3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

Simplify:

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

(b) Evaluate the derivative at $(3, 3)$:

Substitute $x = 3$, $y = 3$:

$$\frac{dy}{dx} = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6 - 9}{9 - 6} = \frac{-3}{3} = \boxed{-1}$$

Question 3

Consider the equation:

$$x^2 \sin(y) + y^2 \cos(x) = 1$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

Question 3 - Solution

We are given:

$$x^2 \sin(y) + y^2 \cos(x) = 1$$

Differentiate both sides implicitly with respect to x :

$$\frac{d}{dx} [x^2 \sin(y)] + \frac{d}{dx} [y^2 \cos(x)] = 0$$

Using the product rule and chain rule:

$$2x \sin(y) + x^2 \cos(y) \frac{dy}{dx} + 2y \frac{dy}{dx} \cos(x) - y^2 \sin(x) = 0$$

Move the non- $\frac{dy}{dx}$ terms to the other side:

$$x^2 \cos(y) \frac{dy}{dx} + 2y \cos(x) \frac{dy}{dx} = y^2 \sin(x) - 2x \sin(y)$$

Factor out $\frac{dy}{dx}$:

$$(x^2 \cos(y) + 2y \cos(x)) \frac{dy}{dx} = y^2 \sin(x) - 2x \sin(y)$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y^2 \sin(x) - 2x \sin(y)}{x^2 \cos(y) + 2y \cos(x)}$$

At $(0, 1)$:

$$\frac{dy}{dx} = \frac{(1)^2 \sin(0) - 2(0) \sin(1)}{(0)^2 \cos(1) + 2(1) \cos(0)} = \frac{0}{2} = \boxed{0}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{0}$$

Question 4

Consider the equation:

$$\sin(xy) = x + y$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 0)$.

Question 4 - Solution**(b) Tangent slope**

At $(0, 0)$, the original equation reads $\sin 0 = 0 + 0$, so the point belongs to the curve.

(a) Implicit derivative

Implicit differentiation gives

$$\cos(xy)(y + xy') = 1 + y'.$$

Collecting derivative terms,

$$(x \cos(xy) - 1)y' = 1 - y \cos(xy).$$

Thus, wherever the denominator is nonzero,

$$y' = \frac{1 - y \cos(xy)}{x \cos(xy) - 1}.$$

At the specified point the denominator is -1 , and the slope is $\boxed{-1}$.

Question 5

Consider the equation:

$$x^2y + e^{xy} = y^3 + 4x$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

Question 5 - Solution

We are given:

$$x^2y + e^{xy} = y^3 + 4x$$

Differentiate both sides implicitly with respect to x :

$$\frac{d}{dx}[x^2y] + \frac{d}{dx}[e^{xy}] = \frac{d}{dx}[y^3] + \frac{d}{dx}[4x]$$

Using the product rule and chain rule:

$$x^2 \frac{dy}{dx} + 2xy + e^{xy} \left(x \frac{dy}{dx} + y \right) = 3y^2 \frac{dy}{dx} + 4$$

Expand:

$$x^2 \frac{dy}{dx} + 2xy + xe^{xy} \frac{dy}{dx} + ye^{xy} = 3y^2 \frac{dy}{dx} + 4$$

Move the $\frac{dy}{dx}$ terms to one side:

$$x^2 \frac{dy}{dx} + xe^{xy} \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 4 - 2xy - ye^{xy}$$

Factor:

$$(x^2 + xe^{xy} - 3y^2) \frac{dy}{dx} = 4 - 2xy - ye^{xy}$$

Therefore,

$$\boxed{\frac{dy}{dx} = \frac{4 - 2xy - ye^{xy}}{x^2 + xe^{xy} - 3y^2}}$$

At $(0, 1)$, we have $xy = 0$, so $e^{xy} = e^0 = 1$. Thus,

$$\frac{dy}{dx} = \frac{4 - 2(0)(1) - (1)(1)}{0^2 + 0(1) - 3(1)^2} = \frac{3}{-3} = \boxed{-1}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{-1}$$

Question 6

Consider the equation defined implicitly:

$$x^2 + y^2 + \ln(y) = 10$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line at the point $(3, 1)$.

Question 6 - Solution

The logarithm requires $y > 0$. Differentiating gives

$$2x + 2yy' + y'/y = 0, \quad y' = \frac{-2x}{2y + 1/y}.$$

At $(3, 1)$, $3^2 + 1^2 + \ln 1 = 10$, so the point is on the curve.

The denominator is 3, giving $y'(3) = -2$.

Question 7

Given the curve defined implicitly by the equation:

$$\sin(xy) + x^2 = y$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the value of $\frac{dy}{dx}$ at the point $(0, 0)$.

Question 7 - Solution

We are given:

$$\sin(xy) + x^2 = y$$

(a) Differentiate both sides implicitly with respect to x :

Use the chain rule on $\sin(xy)$, noting that xy is a product of functions:

$$\frac{d}{dx}[\sin(xy)] + \frac{d}{dx}[x^2] = \frac{d}{dx}[y]$$

Differentiate:

$$\cos(xy) \cdot \left(y + x \cdot \frac{dy}{dx} \right) + 2x = \frac{dy}{dx}$$

Now solve for $\frac{dy}{dx}$. Expand the left-hand side:

$$\cos(xy)y + \cos(xy)x \cdot \frac{dy}{dx} + 2x = \frac{dy}{dx}$$

Bring all $\frac{dy}{dx}$ terms to one side:

$$\cos(xy)x \cdot \frac{dy}{dx} - \frac{dy}{dx} = -\cos(xy)y - 2x$$

Factor $\frac{dy}{dx}$:

$$(\cos(xy)x - 1) \cdot \frac{dy}{dx} = -\cos(xy)y - 2x$$

Solve:

$$\boxed{\frac{dy}{dx} = \frac{-\cos(xy)y - 2x}{\cos(xy)x - 1}}$$

(b) Evaluate at the point $(0, 0)$:

Substitute $x = 0$, $y = 0$ into the derivative:

$$\cos(0) = 1$$

$$\frac{dy}{dx} = \frac{-1(0) - 2(0)}{1(0) - 1} = \frac{0}{-1} = 0$$

$$\boxed{\left. \frac{dy}{dx} \right|_{(0,0)} = 0}$$

Question 8

Consider the equation:

$$\ln(x + y) + x^2y = y^2 + 3x + \ln 2 - 3$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(1, 1)$.

Question 8 - Solution

At $(1, 1)$ both sides of the corrected equation equal $\ln 2 + 1$.

(a) Implicit derivative

Differentiating (with $x + y > 0$) gives

$$\frac{1 + y'}{x + y} + 2xy + x^2y' = 2yy' + 3.$$

Therefore

$$y' = \frac{3 - 2xy - 1/(x + y)}{1/(x + y) + x^2 - 2y}.$$

(b) Tangent slope

At $(1, 1)$ the numerator is $1/2$ and the denominator is $-1/2$, hence the slope is $\boxed{-1}$.

Question 9

Consider the equation:

$$x \sin(y) + y \cos(x) = 1$$

- (a) Use implicit differentiation to find $\frac{dy}{dx}$.
- (b) Find the slope of the tangent line to the curve at the point $(0, 1)$.

Question 9 - Solution

We are given:

$$x \sin(y) + y \cos(x) = 1$$

Differentiate both sides implicitly with respect to x :

$$\frac{d}{dx}[x \sin(y)] + \frac{d}{dx}[y \cos(x)] = 0$$

Using the product rule:

$$\sin(y) + x \cos(y) \frac{dy}{dx} + \cos(x) \frac{dy}{dx} - y \sin(x) = 0$$

Move the non- $\frac{dy}{dx}$ terms to the other side:

$$x \cos(y) \frac{dy}{dx} + \cos(x) \frac{dy}{dx} = y \sin(x) - \sin(y)$$

Factor out $\frac{dy}{dx}$:

$$(x \cos(y) + \cos(x)) \frac{dy}{dx} = y \sin(x) - \sin(y)$$

Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{y \sin(x) - \sin(y)}{x \cos(y) + \cos(x)}}$$

At $(0, 1)$:

$$\frac{dy}{dx} = \frac{(1) \sin(0) - \sin(1)}{(0) \cos(1) + \cos(0)}$$

$$\frac{dy}{dx} = \frac{0 - \sin(1)}{0 + 1} = \boxed{-\sin(1)}$$

Therefore, the slope of the tangent line at $(0, 1)$ is:

$$\boxed{-\sin(1)}$$

Question 10

Consider the real relation

$$x^2 + y^2 = \sin(xy).$$

- (a) Derive the formal relation obtained by implicit differentiation. State when division would be valid.
- (b) Determine the real solution set and decide whether a tangent slope exists at $(0, 0)$.

Question 10 - Solution

Formal differentiation. If a differentiable branch existed, it would obey

$$2x + 2yy' = \cos(xy)(y + xy'),$$

so $(2y - x \cos(xy))y' = y \cos(xy) - 2x$. Division would require a nonzero coefficient.

Analyze the real point set first. For every real x, y ,

$$\sin(xy) \leq |xy| \leq \frac{x^2 + y^2}{2}.$$

Thus the given equation implies

$$0 \leq x^2 + y^2 \leq \frac{x^2 + y^2}{2},$$

which forces $x = y = 0$. The real solution set is the singleton $\boxed{\{(0, 0)\}}$.

There is no curve branch through this isolated point and hence no tangent slope dy/dx there. The formal expression $0/0$ alone would not establish this conclusion.