

Question 1

Let

$$f(x) = \frac{1}{x+2}$$

Use the definition of the derivative to compute $f'(1)$. Show all steps.

Question 1 - Solution

We are given:

$$f(x) = \frac{1}{x+2}$$

Using the definition of the derivative:

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

Compute:

$$f(1+h) = \frac{1}{(1+h)+2} = \frac{1}{h+3}, \quad f(1) = \frac{1}{1+2} = \frac{1}{3}$$

So:

$$f'(1) = \lim_{h \rightarrow 0} \frac{\frac{1}{h+3} - \frac{1}{3}}{h}$$

Find a common denominator for the numerator:

$$\frac{1}{h+3} - \frac{1}{3} = \frac{3 - (h+3)}{3(h+3)} = \frac{-h}{3(h+3)}$$

Now plug back into the limit:

$$f'(1) = \lim_{h \rightarrow 0} \frac{-h}{3(h+3)} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} \frac{-1}{3(h+3)}$$

Now evaluate the limit:

$$f'(1) = \frac{-1}{3(0+3)} = \boxed{-\frac{1}{9}}$$

Question 2

Let

$$f(x) = \sqrt{4x + 1}$$

Use the definition of the derivative to compute $f'(2)$. Show all steps clearly without using shortcut rules.

Question 2 - Solution

We are given:

$$f(x) = \sqrt{4x + 1}$$

Using the limit definition of the derivative:

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

Compute:

$$f(2+h) = \sqrt{4(2+h) + 1} = \sqrt{8+4h+1} = \sqrt{4h+9}$$

$$f(2) = \sqrt{4(2) + 1} = \sqrt{9} = 3$$

So,

$$f'(2) = \lim_{h \rightarrow 0} \frac{\sqrt{4h+9} - 3}{h}$$

To evaluate this limit, multiply numerator and denominator by the conjugate of the numerator:

$$f'(2) = \lim_{h \rightarrow 0} \frac{\sqrt{4h+9} - 3}{h} \cdot \frac{\sqrt{4h+9} + 3}{\sqrt{4h+9} + 3}$$

Apply difference of squares in the numerator:

$$= \lim_{h \rightarrow 0} \frac{(4h+9) - 9}{h(\sqrt{4h+9} + 3)} = \lim_{h \rightarrow 0} \frac{4h}{h(\sqrt{4h+9} + 3)}$$

Cancel h in numerator and denominator:

$$= \lim_{h \rightarrow 0} \frac{4}{\sqrt{4h+9} + 3}$$

Now evaluate the limit:

$$f'(2) = \frac{4}{\sqrt{9+3}} = \frac{4}{3+3} = \boxed{\frac{2}{3}}$$

Question 3

Let

$$f(x) = \frac{1}{x+3}$$

Use the definition of the derivative to compute $f'(1)$. Show all steps clearly without using differentiation shortcuts.

Question 3 - Solution

We are given:

$$f(x) = \frac{1}{x+3}$$

Using the definition of the derivative:

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

Compute:

$$f(1+h) = \frac{1}{1+h+3} = \frac{1}{h+4}, \quad f(1) = \frac{1}{4}$$

So,

$$f'(1) = \lim_{h \rightarrow 0} \frac{\frac{1}{h+4} - \frac{1}{4}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h \rightarrow 0} \frac{\frac{4-(h+4)}{4(h+4)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{4(h+4)}}{h}$$

Simplify:

$$= \lim_{h \rightarrow 0} \frac{-1}{4(h+4)}$$

Now take the limit:

$$f'(1) = \frac{-1}{4(4)} = \boxed{-\frac{1}{16}}$$

Question 4

Let

$$f(x) = x^2$$

Use the definition of the derivative to compute $f'(3)$. Do not use differentiation rules.

Question 4 - Solution

We are given:

$$f(x) = x^2$$

Using the definition of the derivative at $x = 3$:

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

Compute each term:

$$f(3+h) = (3+h)^2 = 9 + 6h + h^2$$

$$f(3) = 3^2 = 9$$

Substitute into the limit:

$$f'(3) = \lim_{h \rightarrow 0} \frac{(9 + 6h + h^2) - 9}{h} = \lim_{h \rightarrow 0} \frac{6h + h^2}{h}$$

Factor h from the numerator:

$$= \lim_{h \rightarrow 0} \frac{h(6 + h)}{h}$$

Cancel h :

$$= \lim_{h \rightarrow 0} (6 + h)$$

Now take the limit:

$$f'(3) = \boxed{6}$$

Question 5

Let

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative to find an expression for $f'(x)$. Then evaluate $f'(2)$ using your expression.

Question 5 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h}$$

Simplify:

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}$$

Take the limit:

$$f'(x) = \frac{-1}{x^2}$$

Now evaluate at $x = 2$:

$$f'(2) = \frac{-1}{2^2} = \frac{-1}{4}$$

Final Answer:

$$f'(x) = \frac{-1}{x^2}, \quad f'(2) = \boxed{-\frac{1}{4}}$$

Question 6

Let

$$f(x) = \sqrt{x+1}$$

Use the definition of the derivative to find $f'(x)$. Show all steps clearly without using derivative shortcuts.

Question 6 - Solution

We are given:

$$f(x) = \sqrt{x+1}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

To simplify the numerator, multiply by the conjugate:

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

This becomes:

$$= \lim_{h \rightarrow 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

Cancel the h 's:

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x+1}}}$$

Question 7

Let

$$f(x) = \frac{1}{x}$$

Use the definition of the derivative to compute $f'(x)$. Show all steps clearly without using derivative rules.

Question 7 - Solution

We are given:

$$f(x) = \frac{1}{x}$$

Use the definition:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Combine the terms in the numerator:

$$= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h}$$

Simplify:

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{-1}{x^2}$$

Final Answer:

$$f'(x) = \boxed{-\frac{1}{x^2}}$$

Question 8

Let

$$f(x) = \sqrt{x}$$

Use the definition of the derivative to compute $f'(x)$. Do not use derivative shortcuts. Show all algebraic simplification and steps clearly.

Question 8 - Solution

We are given:

$$f(x) = \sqrt{x}$$

Use the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Multiply numerator and denominator by the conjugate:

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

Simplify numerator:

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x}}}$$

Question 9

Let

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Task: Use the *definition of the derivative* to find $f'(x)$.

- Use first principles only.
- Do not apply derivative shortcuts.
- Show all algebraic steps.

Question 9 - Solution

We are given:

$$f(x) = \sqrt{x^2 + 4x + 5}$$

Apply the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2 + 4(x+h) + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Expand the expression inside the first square root:

$$(x+h)^2 + 4(x+h) + 5 = x^2 + 2xh + h^2 + 4x + 4h + 5$$

So we have:

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h}$$

Multiply numerator and denominator by the conjugate:

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} - \sqrt{x^2 + 4x + 5}}{h} \cdot \frac{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Use the identity $(a-b)(a+b) = a^2 - b^2$:

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 + 4x + 4h + 5) - (x^2 + 4x + 5)}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Simplify the numerator:

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 4h}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Factor h from the numerator:

$$= \lim_{h \rightarrow 0} \frac{h(2x + h + 4)}{h (\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5})}$$

Cancel h :

$$= \lim_{h \rightarrow 0} \frac{2x + h + 4}{\sqrt{x^2 + 2xh + h^2 + 4x + 4h + 5} + \sqrt{x^2 + 4x + 5}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}$$

Final Answer:

$$f'(x) = \boxed{\frac{2x + 4}{2\sqrt{x^2 + 4x + 5}}}$$

Question 10

Let

$$f(x) = \sqrt{x+3}$$

Use the definition of the derivative to find $f'(x)$. Do not apply any derivative shortcuts — use first principles only.

Question 10 - Solution

We are given:

$$f(x) = \sqrt{x+3}$$

Apply the definition of the derivative:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

To evaluate this limit, multiply numerator and denominator by the conjugate of the numerator:

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}}$$

Use the identity $(a-b)(a+b) = a^2 - b^2$:

$$= \lim_{h \rightarrow 0} \frac{(x+h+3) - (x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

Cancel h in numerator and denominator:

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

Now take the limit as $h \rightarrow 0$:

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

Final Answer:

$$f'(x) = \boxed{\frac{1}{2\sqrt{x+3}}}$$