

Question 1

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

Find the values of constants k and c such that $f(x)$ is continuous and differentiable at $x = 2$.

Question 1 - Solution

We are given:

$$f(x) = \begin{cases} kx^2 + 3x + 1, & \text{if } x < 2 \\ x^3 - 3kx + c, & \text{if } x \geq 2 \end{cases}$$

We want $f(x)$ to be both:

1. Continuous at $x = 2$
2. Differentiable at $x = 2$

Step 1: Continuity at $x = 2$

For continuity, we need:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

Left-hand limit:

$$\lim_{x \rightarrow 2^-} f(x) = k(2)^2 + 3(2) + 1 = 4k + 6 + 1 = 4k + 7$$

Right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = (2)^3 - 3k(2) + c = 8 - 6k + c$$

Set them equal:

$$4k + 7 = 8 - 6k + c \quad (1)$$

Step 2: Differentiability at $x = 2$

Derivatives:

$$\text{Left: } f'(x) = 2kx + 3 \Rightarrow f'(2^-) = 4k + 3$$

$$\text{Right: } f'(x) = 3x^2 - 3k \Rightarrow f'(2^+) = 12 - 3k$$

Set equal:

$$4k + 3 = 12 - 3k \quad (2)$$

Step 3: Solve the system

From (2):

$$4k + 3 = 12 - 3k \Rightarrow 7k = 9 \Rightarrow k = \frac{9}{7}$$

Substitute into (1):

$$4\left(\frac{9}{7}\right) + 7 = 8 - 6\left(\frac{9}{7}\right) + c \Rightarrow \frac{36}{7} + 7 = 8 - \frac{54}{7} + c \Rightarrow \frac{85}{7} = \frac{2}{7} + c \Rightarrow c = \frac{83}{7}$$

Final Answer:

$$\boxed{k = \frac{9}{7}, \quad c = \frac{83}{7}}$$

Question 2

For real constants a and b , let

$$f(x) = \begin{cases} \frac{\sin(ax)}{x}, & x \neq 0, \\ b, & x = 0. \end{cases}$$

Classify the pairs (a, b) for which f is continuous at zero and those for which it has a removable discontinuity there.

State how to remove the discontinuity.

Question 2 - Solution

For $a \neq 0$, write

$$\frac{\sin(ax)}{x} = a \frac{\sin(ax)}{ax} \longrightarrow a.$$

For $a = 0$, the quotient is identically zero away from zero, so the same limit holds.

Thus $\lim_{x \rightarrow 0} f(x) = a$ for every real a .

Since $f(0) = b$, continuity holds exactly when $\boxed{b = a}$.

When $\boxed{b \neq a}$ the limit exists but differs from $f(0)$, so the discontinuity is removable by resetting $f(0) = a$.

A function cannot be both continuous and discontinuous at the same point.

Question 3

Let the function $f(x)$ be defined by:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Determine the value of k such that $f(x)$ is continuous at $x = 2$.

Question 3 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

To make $f(x)$ continuous at $x = 2$, the following condition must hold:

$$\lim_{x \rightarrow 2} f(x) = f(2)$$

Step 1: Compute the limit as $x \rightarrow 2$

Factor the numerator:

$$x^2 - 4 = (x - 2)(x + 2)$$

For $x \neq 2$, the function simplifies to:

$$f(x) = x + 2$$

So,

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x + 2) = 4$$

Step 2: Match the function value

Continuity requires:

$$f(2) = k = 4$$

Final Answer:

$$\boxed{k = 4}$$

Question 4

Consider the function

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

Find the value of c that makes $f(x)$ continuous at $x = 0$.

Question 4 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

We want $f(x)$ to be continuous at $x = 0$. This means:

$$\lim_{x \rightarrow 0} f(x) = f(0) = c$$

Step 1: Compute the limit as $x \rightarrow 0$

We use the standard limit identity:

$$\lim_{x \rightarrow 0} \frac{\sin(kx)}{x} = k$$

So, in our case:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3$$

Step 2: Set the limit equal to the function value

To ensure continuity at $x = 0$, we must have:

$$c = 3$$

Final Answer:

$$\boxed{c = 3}$$

Question 5

Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} kx^2 + 2x, & x < 3 \\ x^2 - 4kx + 9, & x \geq 3 \end{cases}$$

Find the value of k that makes $f(x)$ continuous at $x = 3$.

Question 5 - Solution

We are given:

$$f(x) = \begin{cases} kx^2 + 2x, & x < 3 \\ x^2 - 4kx + 9, & x \geq 3 \end{cases}$$

To ensure continuity at $x = 3$, the left-hand and right-hand limits must agree, and they must equal the function value:

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

Step 1: Left-hand limit as $x \rightarrow 3^-$

From the first piece:

$$\lim_{x \rightarrow 3^-} f(x) = k(3)^2 + 2(3) = 9k + 6$$

Step 2: Right-hand limit and $f(3)$

From the second piece:

$$f(3) = (3)^2 - 4k(3) + 9 = 9 - 12k + 9 = 18 - 12k$$

Step 3: Set the two expressions equal for continuity

$$9k + 6 = 18 - 12k$$

Solve for k :

$$9k + 12k = 18 - 6 \quad 21k = 12 \quad k = \frac{12}{21} = \frac{4}{7}$$

Final Answer:

$$\boxed{k = \frac{4}{7}}$$

Question 6

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{\sin(ax)}{x}, & x \neq 0 \\ b, & x = 0 \end{cases}$$

Find the values of constants a and b such that $f(x)$ is continuous at $x = 0$.

Question 6 - Solution

To ensure continuity at $x = 0$, we must have:

$$\lim_{x \rightarrow 0} f(x) = f(0) = b$$

We are given:

$$f(x) = \frac{\sin(ax)}{x}, \quad x \neq 0$$
$$f(0) = b$$

We evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(ax)}{x} = \lim_{x \rightarrow 0} a \cdot \frac{\sin(ax)}{ax} = a \cdot \lim_{u \rightarrow 0} \frac{\sin u}{u} = a \cdot 1 = a$$

So for continuity, we must have:

$$\lim_{x \rightarrow 0} f(x) = a = b$$

Final Answer:

Any $a \in \mathbb{R}$ and $b = a$

Question 7

Let the function $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 - 4, & x < 2 \\ kx + 1, & x \geq 2 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 2$.

Question 7 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} x^2 - 4, & x < 2 \\ kx + 1, & x \geq 2 \end{cases}$$

To ensure continuity at $x = 2$, we require:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

Evaluate the left-hand limit:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 - 4) = 4 - 4 = 0$$

Evaluate the right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (kx + 1) = 2k + 1$$

Set the two limits equal for continuity:

$$2k + 1 = 0 \quad \Rightarrow \quad k = -\frac{1}{2}$$

Final Answer:

$$\boxed{k = -\frac{1}{2}}$$

Question 8

Let

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

Find the value of c that makes $f(x)$ continuous at $x = 0$.

Question 8 - Solution

To ensure continuity at $x = 0$, we require:

$$\lim_{x \rightarrow 0} f(x) = f(0) = c$$

Compute the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} \frac{3 \sin(3x)}{3x} = 3 \cdot \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3$$

Thus, to make f continuous at $x = 0$, we must set:

$$\boxed{c = 3}$$

Question 9

Let

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

Determine the value of k such that $f(x)$ is continuous at $x = 2$.

Question 9 - Solution

We want f to be continuous at $x = 2$, so we must have:

$$\lim_{x \rightarrow 2} f(x) = f(2) = k$$

First, compute the limit:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$$

For $x \neq 2$, we can cancel the common factor:

$$f(x) = x + 2$$

Therefore:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x + 2) = 4$$

To make f continuous at $x = 2$, we set:

$$\boxed{k = 4}$$

Question 10

Let

$$f(x) = \begin{cases} \frac{\sin(3x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

- (a) Find the value of a such that $f(x)$ is continuous at $x = 0$.
- (b) Is $f(x)$ differentiable at $x = 0$? Justify your answer.

Question 10 - Solution

(a) For continuity at $x = 0$, we must have

$$\lim_{x \rightarrow 0} f(x) = f(0) = a.$$

We compute the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} 3 \cdot \frac{\sin(3x)}{3x} = 3 \cdot 1 = 3.$$

Thus, to make $f(x)$ continuous at $x = 0$, we must set

$$\boxed{a = 3}.$$

(b) Differentiability at $x = 0$:

We use the definition of the derivative:

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(3h)}{h} - 3}{h}.$$

Rewrite the expression:

$$\frac{\frac{\sin(3h)}{h} - 3}{h} = \frac{\sin(3h) - 3h}{h^2}.$$

Rescale by $u = 3h$ (so $u^2 = 9h^2$):

$$\frac{\sin(3h) - 3h}{h^2} = 9 \cdot \frac{\sin(3h) - 3h}{(3h)^2}.$$

Let $u = 3h$. Then as $h \rightarrow 0$, $u \rightarrow 0$, and the limit becomes

$$f'(0) = 9 \lim_{u \rightarrow 0} \frac{\sin u - u}{u^2}.$$

Using the standard limit

$$\lim_{u \rightarrow 0} \frac{\sin u - u}{u^2} = 0,$$

we conclude that

$$f'(0) = 0.$$

Conclusion: The limit exists, so $f(x)$ is differentiable at $x = 0$, and

$$\boxed{f'(0) = 0}.$$