

Question 1

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x + 7} - \sqrt{x^2 + x} \right).$$

Question 1 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x + 7} - \sqrt{x^2 + x} \right).$$

This expression has the indeterminate form $\infty - \infty$. To resolve it, we multiply by the conjugate.

Step 1: Multiply by the Conjugate

$$= \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 5x + 7} - \sqrt{x^2 + x} \right) \cdot \frac{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}.$$

This simplifies to

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 5x + 7) - (x^2 + x)}{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}.$$

Step 2: Simplify the Numerator

$$(x^2 + 5x + 7) - (x^2 + x) = 4x + 7,$$

so the limit becomes

$$\lim_{x \rightarrow \infty} \frac{4x + 7}{\sqrt{x^2 + 5x + 7} + \sqrt{x^2 + x}}.$$

Step 3: Factor Out x

Factor x from each square root:

$$= \lim_{x \rightarrow \infty} \frac{4x + 7}{x \left(\sqrt{1 + \frac{5}{x} + \frac{7}{x^2}} + \sqrt{1 + \frac{1}{x}} \right)}.$$

Divide numerator and denominator by x :

$$= \lim_{x \rightarrow \infty} \frac{4 + \frac{7}{x}}{\sqrt{1 + \frac{5}{x} + \frac{7}{x^2}} + \sqrt{1 + \frac{1}{x}}}.$$

Step 4: Take the Limit

As $x \rightarrow \infty$,

$$\frac{7}{x} \rightarrow 0, \quad \frac{5}{x} \rightarrow 0, \quad \frac{7}{x^2} \rightarrow 0, \quad \frac{1}{x} \rightarrow 0.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{7}{x}}{\sqrt{1 + \frac{5}{x} + \frac{7}{x^2}} + \sqrt{1 + \frac{1}{x}}} = \frac{4}{1 + 1} = 2.$$

Final Answer

2

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 7x - 5}{\sqrt{x^4 + 2x^2 + 1}}.$$

Question 2 - Solution

We evaluate

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 7x - 5}{\sqrt{x^4 + 2x^2 + 1}}.$$

Step 1: Rewrite the denominator carefully

Note that

$$\sqrt{x^4 + 2x^2 + 1} = \sqrt{(x^2 + 1)^2} = |x^2 + 1|.$$

Since $x^2 + 1 > 0$ for all real x ,

$$|x^2 + 1| = x^2 + 1.$$

Thus, the expression becomes

$$\frac{3x^2 + 7x - 5}{x^2 + 1}.$$

Step 2: Divide by the highest power of x

Divide both numerator and denominator by x^2 :

$$\frac{3 + \frac{7}{x} - \frac{5}{x^2}}{1 + \frac{1}{x^2}}.$$

Step 3: Take the limit

As $x \rightarrow -\infty$,

$$\frac{7}{x} \rightarrow 0, \quad \frac{5}{x^2} \rightarrow 0, \quad \frac{1}{x^2} \rightarrow 0.$$

Therefore,

$$\lim_{x \rightarrow -\infty} \frac{3 + \frac{7}{x} - \frac{5}{x^2}}{1 + \frac{1}{x^2}} = \frac{3}{1} = 3.$$

Final Answer

$$\boxed{3}$$

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 6x + 5} - \sqrt{x^2 + 2x + 1} \right).$$

Question 3 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 6x + 5} - \sqrt{x^2 + 2x + 1} \right).$$

This expression has the indeterminate form $\infty - \infty$, so we rationalize.

Step 1: Multiply by the Conjugate

$$= \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 6x + 5} - \sqrt{x^2 + 2x + 1} \right) \cdot \frac{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}.$$

This simplifies to

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 6x + 5) - (x^2 + 2x + 1)}{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}.$$

Step 2: Simplify the Numerator

$$(x^2 + 6x + 5) - (x^2 + 2x + 1) = 4x + 4,$$

so the limit becomes

$$\lim_{x \rightarrow \infty} \frac{4x + 4}{\sqrt{x^2 + 6x + 5} + \sqrt{x^2 + 2x + 1}}.$$

Step 3: Factor Out x

Factor x from each square root:

$$= \lim_{x \rightarrow \infty} \frac{4x + 4}{x \left(\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{2}{x} + \frac{1}{x^2}} \right)}.$$

Divide numerator and denominator by x :

$$= \lim_{x \rightarrow \infty} \frac{4 + \frac{4}{x}}{\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{2}{x} + \frac{1}{x^2}}}.$$

Step 4: Take the Limit

As $x \rightarrow \infty$, all terms involving $\frac{1}{x}$ and higher powers go to 0. Therefore,

$$\lim_{x \rightarrow \infty} \frac{4 + \frac{4}{x}}{\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}} + \sqrt{1 + \frac{2}{x} + \frac{1}{x^2}}} = \frac{4}{1 + 1} = 2.$$

Final Answer

2

Question 4

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 2x^2 - 5x + 7}{x^3 - 4x + 1}$$

Question 4 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 2x^2 - 5x + 7}{x^3 - 4x + 1}$$

Step 1: Compare the degrees of numerator and denominator

Both numerator and denominator are degree 3 polynomials.

When the degrees are the same, the limit as $x \rightarrow \pm\infty$ is the ratio of the leading coefficients.

Step 2: Extract leading terms

Numerator leading term: x^3 , Denominator leading term: x^3

So,

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 2x^2 - 5x + 7}{x^3 - 4x + 1} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{2}{x} - \frac{5}{x^2} + \frac{7}{x^3}}{1 - \frac{4}{x^2} + \frac{1}{x^3}}$$

As $x \rightarrow -\infty$, all terms with $\frac{1}{x^k} \rightarrow 0$ for $k > 0$

Step 3: Evaluate the limit

$$\lim_{x \rightarrow -\infty} \frac{1 + 0 - 0 + 0}{1 - 0 + 0} = \frac{1}{1} = 1$$

Final Answer:

$$\boxed{1}$$

Question 5

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 3x - 7}{\sqrt{x^4 + 2x^3 + 5}} \right)$$

Question 5 - Solution

We are given:

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 3x - 7}{\sqrt{x^4 + 2x^3 + 5}} \right)$$

Step 1: Understand the growth of numerator and denominator

As $x \rightarrow \infty$, both numerator and denominator tend to infinity, so we need to simplify.

Step 2: Factor out highest power of x

In the numerator: highest power is x^2 In the denominator (inside square root): highest power is x^4

Factor numerator:

$$x^2 \left(1 + \frac{3}{x} - \frac{7}{x^2} \right)$$

Factor denominator inside square root:

$$\sqrt{x^4 \left(1 + \frac{2}{x} + \frac{5}{x^4} \right)} = x^2 \sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}$$

Step 3: Substitute factored forms into the expression**

$$\frac{x^2 \left(1 + \frac{3}{x} - \frac{7}{x^2} \right)}{x^2 \sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}}$$

Cancel x^2 :

$$\frac{1 + \frac{3}{x} - \frac{7}{x^2}}{\sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}}$$

Step 4: Take the limit as $x \rightarrow \infty$

As $x \rightarrow \infty$, all terms with $\frac{1}{x}$ go to 0:

$$\lim_{x \rightarrow \infty} \frac{1 + \frac{3}{x} - \frac{7}{x^2}}{\sqrt{1 + \frac{2}{x} + \frac{5}{x^4}}} = \frac{1 + 0 - 0}{\sqrt{1 + 0 + 0}} = \frac{1}{1} = 1$$

Final Answer:

$$\boxed{1}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \left(\frac{3x^3 + 2x^2 - x + 5}{\sqrt{x^6 + x^2 + 1}} \right)$$

Question 6 - Solution

We are given:

$$\lim_{x \rightarrow -\infty} \left(\frac{3x^3 + 2x^2 - x + 5}{\sqrt{x^6 + x^2 + 1}} \right)$$

Step 1: Factor out highest powers

Numerator is a cubic polynomial: factor x^3 :

$$3x^3 + 2x^2 - x + 5 = x^3 \left(3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3} \right)$$

Denominator is a square root of a degree-6 polynomial:

$$\sqrt{x^6 + x^2 + 1} = |x^3| \sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}$$

Note: Since $x \rightarrow -\infty$, we have $|x^3| = -x^3$

So,

$$\sqrt{x^6 + x^2 + 1} = -x^3 \sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}$$

Step 2: Substitute into the expression

$$\frac{x^3 \left(3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3} \right)}{-x^3 \sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}}$$

Cancel x^3 :

$$\frac{3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3}}{-\sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}}$$

Step 3: Take the limit as $x \rightarrow -\infty$

All terms with $\frac{1}{x^n} \rightarrow 0$. So:

$$\lim_{x \rightarrow -\infty} \frac{3 + \frac{2}{x} - \frac{1}{x^2} + \frac{5}{x^3}}{-\sqrt{1 + \frac{1}{x^4} + \frac{1}{x^6}}} = \frac{3}{-1} = -3$$

Final Answer:

$$\boxed{-3}$$

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 5x} - x \right).$$

Question 7 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 5x} - x \right).$$

At first glance, as $x \rightarrow \infty$,

$$\sqrt{x^2 + 5x} \rightarrow \infty \quad \text{and} \quad x \rightarrow \infty,$$

so the expression inside the parentheses has the indeterminate form $\infty - \infty$.

Step 1: Rationalize the Expression

Multiply the expression inside the parentheses by its conjugate:

$$\sqrt{x^2 + 5x} - x = \frac{(\sqrt{x^2 + 5x} - x)(\sqrt{x^2 + 5x} + x)}{\sqrt{x^2 + 5x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, we obtain

$$= \frac{(x^2 + 5x) - x^2}{\sqrt{x^2 + 5x} + x} = \frac{5x}{\sqrt{x^2 + 5x} + x}.$$

Thus the original limit becomes

$$\lim_{x \rightarrow \infty} x \cdot \frac{5x}{\sqrt{x^2 + 5x} + x}.$$

Step 2: Simplify

Rewrite the expression as

$$\lim_{x \rightarrow \infty} \frac{5x^2}{\sqrt{x^2 + 5x} + x}.$$

Factor x out of the square root:

$$\sqrt{x^2 + 5x} = x\sqrt{1 + \frac{5}{x}} \quad (x > 0).$$

Substitute:

$$\frac{5x^2}{x \left(\sqrt{1 + \frac{5}{x}} + 1 \right)} = \frac{5x}{\sqrt{1 + \frac{5}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{5}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{5}{x}} \rightarrow 1.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{5x}{\sqrt{1 + \frac{5}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{5x}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .

Question 8

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x^2 \left(\sqrt{x^2 + 4x} - x \right).$$

Question 8 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x^2 \left(\sqrt{x^2 + 4x} - x \right).$$

As $x \rightarrow \infty$, both $\sqrt{x^2 + 4x}$ and x approach infinity, so the expression inside the parentheses has the indeterminate form $\infty - \infty$. Multiplying by x^2 significantly changes the behavior of the limit, so careful analysis is required.

Step 1: Rationalize the Expression

We first rationalize the difference:

$$\sqrt{x^2 + 4x} - x = \frac{(\sqrt{x^2 + 4x} - x)(\sqrt{x^2 + 4x} + x)}{\sqrt{x^2 + 4x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, this becomes

$$= \frac{(x^2 + 4x) - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}.$$

Substitute into the original expression:

$$\lim_{x \rightarrow \infty} x^2 \cdot \frac{4x}{\sqrt{x^2 + 4x} + x} = \lim_{x \rightarrow \infty} \frac{4x^3}{\sqrt{x^2 + 4x} + x}.$$

Step 2: Simplify

Factor x out of the square root:

$$\sqrt{x^2 + 4x} = x\sqrt{1 + \frac{4}{x}} \quad (x > 0).$$

Then

$$\frac{4x^3}{x\left(\sqrt{1 + \frac{4}{x}} + 1\right)} = \frac{4x^2}{\sqrt{1 + \frac{4}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{4}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{4}{x}} \rightarrow 1.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{4x^2}{\sqrt{1 + \frac{4}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{4x^2}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 4x^2 - 7}{2x^3 - 5x + 1}$$

Question 9 - Solution

We are given the rational function:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 4x^2 - 7}{2x^3 - 5x + 1}$$

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Step 1: Analyze degrees of numerator and denominator

- The highest degree in both numerator and denominator is 3. - For limits at infinity, when the degrees match, the limit is the ratio of the leading coefficients.

So:

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 4x^2 - 7}{2x^3 - 5x + 1} = \frac{1}{2}$$

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Step 2: Justify with algebraic manipulation (optional)

Divide numerator and denominator by x^3 :

$$= \frac{1 + \frac{4}{x} - \frac{7}{x^3}}{2 - \frac{5}{x^2} + \frac{1}{x^3}}$$

As $x \rightarrow -\infty$, the fractions go to 0:

$$\frac{1 + 0 - 0}{2 - 0 + 0} = \frac{1}{2}$$

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Final Answer:

$$\boxed{\frac{1}{2}}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 3x} - x \right).$$

Question 10 - Solution

We evaluate

$$\lim_{x \rightarrow \infty} x \left(\sqrt{x^2 + 3x} - x \right).$$

As $x \rightarrow \infty$, both $\sqrt{x^2 + 3x}$ and x grow without bound, so the expression inside the parentheses has the indeterminate form $\infty - \infty$. Multiplying by x changes the rate at which the expression grows, so careful analysis is required.

Step 1: Rationalize the Expression

We first rationalize the difference:

$$\sqrt{x^2 + 3x} - x = \frac{(\sqrt{x^2 + 3x} - x)(\sqrt{x^2 + 3x} + x)}{\sqrt{x^2 + 3x} + x}.$$

Using the identity $(a - b)(a + b) = a^2 - b^2$, this becomes

$$= \frac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x} = \frac{3x}{\sqrt{x^2 + 3x} + x}.$$

Substitute this into the original limit:

$$\lim_{x \rightarrow \infty} x \cdot \frac{3x}{\sqrt{x^2 + 3x} + x} = \lim_{x \rightarrow \infty} \frac{3x^2}{\sqrt{x^2 + 3x} + x}.$$

Step 2: Simplify

Factor x out of the square root:

$$\sqrt{x^2 + 3x} = x\sqrt{1 + \frac{3}{x}} \quad (x > 0).$$

Then

$$\frac{3x^2}{x \left(\sqrt{1 + \frac{3}{x}} + 1 \right)} = \frac{3x}{\sqrt{1 + \frac{3}{x}} + 1}.$$

Step 3: Take the Limit

As $x \rightarrow \infty$, $\frac{3}{x} \rightarrow 0$, so

$$\sqrt{1 + \frac{3}{x}} \rightarrow 1.$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{1 + \frac{3}{x}} + 1} = \lim_{x \rightarrow \infty} \frac{3x}{2} = \infty.$$

Final Answer

The limit diverges to ∞ .