

Question 1

Evaluate the following limit, showing all necessary steps and justification:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x}$$

Question 1 - Solution

We are given:

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x}$$

At $x = 0$, this becomes:

$$\frac{\sqrt{0+4}-2}{0} = \frac{2-2}{0} = \frac{0}{0}$$

This is an indeterminate form, so we must simplify the expression.

Step 1: Multiply numerator and denominator by the conjugate.

Multiply by:

$$\frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2}$$
$$\frac{\sqrt{x^2 + 4} - 2}{x} \cdot \frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2} = \frac{(\sqrt{x^2 + 4} - 2)(\sqrt{x^2 + 4} + 2)}{x(\sqrt{x^2 + 4} + 2)}$$

The numerator is a difference of squares:

$$(\sqrt{x^2 + 4})^2 - 2^2 = x^2 + 4 - 4 = x^2$$

So the entire expression becomes:

$$\frac{x^2}{x(\sqrt{x^2 + 4} + 2)} = \frac{x}{\sqrt{x^2 + 4} + 2}$$

Step 2: Evaluate the limit.

Now:

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x^2 + 4} + 2} = \frac{0}{\sqrt{0+4} + 2} = \frac{0}{4} = 0$$

Final Answer:

$$\boxed{0}$$

Question 2

Evaluate the limit:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

Question 2 - Solution

We are given:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

Step 1: Check for indeterminate form.

At $x = 3$, the numerator becomes:

$$x^2 - 9 = 9 - 9 = 0$$

The denominator becomes:

$$x - 3 = 0$$

So we have the indeterminate form $\frac{0}{0}$. We must simplify.

Step 2: Factor the numerator.

$$x^2 - 9 = (x - 3)(x + 3)$$

$$\Rightarrow \frac{(x - 3)(x + 3)}{x - 3}$$

Cancel the common factor $x - 3$ (note: this is valid for $x \neq 3$):

$$= x + 3$$

Step 3: Take the limit of the simplified expression.

$$\lim_{x \rightarrow 3} x + 3 = 3 + 3 = \boxed{6}$$

Final Answer: $\boxed{6}$

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

Question 3 - Solution

We are given:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

Step 1: Check for indeterminate form.

At $x = 0$, the expression becomes:

$$\frac{\sqrt{1+0} - 1}{0} = \frac{1 - 1}{0} = \frac{0}{0}$$

So we have the indeterminate form $\frac{0}{0}$. To resolve this, we rationalize the numerator.

Step 2: Multiply numerator and denominator by the conjugate of the numerator.

$$\frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \frac{(1+x-1)}{x(\sqrt{1+x}+1)} = \frac{x}{x(\sqrt{1+x}+1)}$$

Cancel x from numerator and denominator:

$$= \frac{1}{\sqrt{1+x} + 1}$$

Step 3: Take the limit of the simplified expression.

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1+1} + 1} = \frac{1}{2}$$

Final Answer:

$$\boxed{\frac{1}{2}}$$

Question 4

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

Question 4 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

Step 1: Use standard limit identity

Recall the standard identity:

$$\lim_{x \rightarrow 0} \frac{\sin(kx)}{x} = k$$

This applies directly here with $k = 3$.

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3$$

Answer: 3

Question 5

Evaluate the following limit:

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

Question 5 - Solution

We are given:

$$\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}$$

Step 1: Direct Substitution

Substitute $x = 4$:

$$\frac{4-4}{\sqrt{4}-2} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify.

Step 2: Use Algebraic Manipulation

We can rationalize the denominator. Multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)}$$

Simplify the denominator:

$$(\sqrt{x}-2)(\sqrt{x}+2) = x-4$$

So the expression becomes:

$$\frac{(x-4)(\sqrt{x}+2)}{x-4}$$

Cancel the common factor $x-4$ (note: valid as $x \rightarrow 4$, not at $x = 4$):

$$= \sqrt{x}+2$$

Now take the limit:

$$\lim_{x \rightarrow 4} \sqrt{x}+2 = \sqrt{4}+2 = 2+2 = \boxed{4}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Question 6 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Step 1: Factor the numerator

Recall that $x^2 - 1 = (x - 1)(x + 1)$. So,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

For $x \neq 1$, we can cancel $x - 1$ from numerator and denominator:

$$= x + 1$$

Step 2: Take the limit

Now that the expression simplifies to $x + 1$, we compute:

$$\lim_{x \rightarrow 1} (x + 1) = 2$$

Answer: 2

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1+x}}{x}.$$

Question 7 - Solution

We evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1+x}}{x}.$$

Step 1: Identify the Indeterminate Form

Substituting $x = 0$ gives

$$\frac{\sqrt{1} - \sqrt{1}}{0} = \frac{0}{0},$$

which is an indeterminate form. Algebraic simplification is required.

Step 2: Multiply by the Conjugate

Multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{1+3x} - \sqrt{1+x}}{x} \cdot \frac{\sqrt{1+3x} + \sqrt{1+x}}{\sqrt{1+3x} + \sqrt{1+x}}.$$

This yields

$$\frac{(1+3x) - (1+x)}{x(\sqrt{1+3x} + \sqrt{1+x})} = \frac{2x}{x(\sqrt{1+3x} + \sqrt{1+x})}.$$

Cancel the common factor of x :

$$\frac{2}{\sqrt{1+3x} + \sqrt{1+x}}.$$

Step 3: Evaluate the Limit

Now substitute $x = 0$:

$$\lim_{x \rightarrow 0} \frac{2}{\sqrt{1+3x} + \sqrt{1+x}} = \frac{2}{1+1} = 1.$$

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Question 8

Evaluate the limit:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x)$$

Question 8 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x} - x)$$

Step 1: Indeterminate Form

As $x \rightarrow \infty$, both $\sqrt{x^2 + 5x}$ and x grow large. The expression becomes $\infty - \infty$, an indeterminate form. We simplify.

Step 2: Multiply by the Conjugate

Multiply numerator and denominator by the conjugate:

$$\begin{aligned} & (\sqrt{x^2 + 5x} - x) \cdot \frac{\sqrt{x^2 + 5x} + x}{\sqrt{x^2 + 5x} + x} \\ &= \frac{(x^2 + 5x) - x^2}{\sqrt{x^2 + 5x} + x} = \frac{5x}{\sqrt{x^2 + 5x} + x} \end{aligned}$$

Step 3: Divide numerator and denominator by x

$$= \frac{5}{\sqrt{1 + \frac{5}{x}} + 1}$$

As $x \rightarrow \infty$, $\frac{5}{x} \rightarrow 0$, so:

$$\lim_{x \rightarrow \infty} \frac{5}{\sqrt{1 + \frac{5}{x}} + 1} = \frac{5}{1 + 1} = \frac{5}{2}$$

Answer:

$$\boxed{\frac{5}{2}}$$

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}.$$

Question 9 - Solution

We evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}.$$

Step 1: Express tangent in terms of sine and cosine

Recall that

$$\tan(5x) = \frac{\sin(5x)}{\cos(5x)}.$$

Thus,

$$\frac{\sin(3x)}{\tan(5x)} = \frac{\sin(3x) \cos(5x)}{\sin(5x)}.$$

Step 2: Apply standard trigonometric limits

Rewrite the expression as

$$\frac{\sin(3x)}{3x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{3}{5} \cdot \cos(5x).$$

Now take limits term by term as $x \rightarrow 0$:

$$\frac{\sin(3x)}{3x} \rightarrow 1, \quad \frac{5x}{\sin(5x)} \rightarrow 1, \quad \cos(5x) \rightarrow 1.$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)} = 1 \cdot 1 \cdot \frac{3}{5} \cdot 1 = \frac{3}{5}.$$

$$\boxed{\frac{3}{5}}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(4x)}{x^2}$$

Question 10 - Solution

We are given the limit:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(4x)}{x^2}$$

Step 1: Use the identity

$$1 - \cos(\theta) = 2 \sin^2\left(\frac{\theta}{2}\right)$$

Apply this identity to the numerator:

$$1 - \cos(4x) = 2 \sin^2(2x)$$

So the expression becomes:

$$\lim_{x \rightarrow 0} \frac{2 \sin^2(2x)}{x^2}$$

Step 2: Rewrite the sine term

$$= \lim_{x \rightarrow 0} 2 \cdot \left(\frac{\sin(2x)}{x}\right)^2$$

We can write:

$$\frac{\sin(2x)}{x} = \frac{\sin(2x)}{2x} \cdot 2 \rightarrow 1 \cdot 2 = 2 \quad \text{as } x \rightarrow 0$$

So:

$$\left(\frac{\sin(2x)}{x}\right)^2 \rightarrow 4 \Rightarrow 2 \cdot 4 = 8$$

Final Answer: 8