

**Question 1**

Let the functions  $f(x)$ ,  $g(x)$ , and  $h(x)$  be defined by the following limits:

$$\lim_{x \rightarrow 2} f(x) = 4, \quad \lim_{x \rightarrow 2} g(x) = -3, \quad \lim_{x \rightarrow 2} h(x) = 0$$

Evaluate the following limit expressions using limit laws and properties:

(a)  $\lim_{x \rightarrow 2} [2f(x) + 5g(x)]$

(b)  $\lim_{x \rightarrow 2} \left[ \frac{f(x) \cdot g(x)}{h(x)} \right]$

(c) Explain whether the given information determines a signed infinite limit in part (b).

Assume the quotient denominator is nonzero in some deleted neighborhood of the approach point. Distinguish finite limits from signed infinite limits.

## Question 1 - Solution

### (a) Sum and constant-multiple laws

Using the given limits,

$$\begin{aligned}\lim_{x \rightarrow 2} [2f(x) + 5g(x)] &= 2 \lim_{x \rightarrow 2} f(x) + 5 \lim_{x \rightarrow 2} g(x) \\ &= 2(4) + 5(-3) = \boxed{-7}.\end{aligned}$$

### (b) Quotient with a zero denominator limit

The product law gives

$$\lim_{x \rightarrow 2} f(x)g(x) = 4(-3) = -12, \quad \lim_{x \rightarrow 2} h(x) = 0.$$

The quotient law does not apply because the denominator tends to zero. Since the numerator tends to a nonzero value, its magnitude stays bounded away from zero near  $x = 2$ . Therefore,

$$\left| \frac{f(x)g(x)}{h(x)} \right| \rightarrow \infty.$$

The quotient has no finite limit.

### (c) Is the sign of an infinite limit determined?

No. Take  $f(x) = 4$  and  $g(x) = -3$ . Each denominator below satisfies the given limit and is nonzero for  $x \neq 2$ :

| Denominator $h(x)$ | Behavior of $-12/h(x)$        |
|--------------------|-------------------------------|
| $(x-2)^2$          | $\rightarrow -\infty$         |
| $-(x-2)^2$         | $\rightarrow +\infty$         |
| $x-2$              | opposite one-sided infinities |

Thus more information about the sign of  $h(x)$  is needed.

A signed infinite limit is not determined.

**Question 2**

Suppose that

$$\lim_{x \rightarrow 3} f(x) = 5 \quad \text{and} \quad \lim_{x \rightarrow 3} g(x) = 0.$$

Define

$$h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

- (a) Explain why direct substitution does not determine  $\lim_{x \rightarrow 3} h(x)$ .
- (b) Use algebraic manipulation to rewrite  $h(x)$ .
- (c) State precisely what additional information would be required to evaluate  $\lim_{x \rightarrow 3} h(x)$ .

**Question 2 - Solution**

We are given:

$$\lim_{x \rightarrow 3} f(x) = 5, \quad \lim_{x \rightarrow 3} g(x) = 0, \quad h(x) = \frac{f(x)^2 - 25}{g(x)}.$$

**(a) Indeterminate form**

Substituting the known limits gives:

$$\lim_{x \rightarrow 3} h(x) = \frac{5^2 - 25}{0} = \frac{0}{0},$$

which is an indeterminate form. Therefore, the limit cannot be evaluated by direct substitution.

**(b) Algebraic simplification**

Factor the numerator:

$$f(x)^2 - 25 = (f(x) - 5)(f(x) + 5).$$

Then,

$$h(x) = \frac{(f(x) - 5)(f(x) + 5)}{g(x)} = \left( \frac{f(x) - 5}{g(x)} \right) (f(x) + 5).$$

Taking limits where possible,

$$\lim_{x \rightarrow 3} (f(x) + 5) = 10.$$

**(c) Required additional information**

The limit

$$\lim_{x \rightarrow 3} h(x)$$

exists if and only if the limit

$$\lim_{x \rightarrow 3} \frac{f(x) - 5}{g(x)}$$

exists.

If we let

$$L = \lim_{x \rightarrow 3} \frac{f(x) - 5}{g(x)},$$

then

$$\lim_{x \rightarrow 3} h(x) = 10L.$$

Without information about how fast  $f(x) \rightarrow 5$  relative to  $g(x) \rightarrow 0$ , the value of the limit cannot be determined.

The limit cannot be evaluated without additional information.

**Question 3**

Given the following limits:

$$\lim_{x \rightarrow 1} f(x) = 4, \quad \lim_{x \rightarrow 1} g(x) = -2, \quad \lim_{x \rightarrow 1} h(x) = 0$$

Define:

$$k(x) = \frac{f(x) \cdot h(x) + g(x)}{h(x)}$$

Evaluate  $\lim_{x \rightarrow 1} k(x)$ , or explain why it does not exist.

Assume the quotient denominator is nonzero in some deleted neighborhood of the approach point. Distinguish finite limits from signed infinite limits.

### Question 3 - Solution

#### Step 1: Apply the product and sum laws

For the numerator,

$$\lim_{x \rightarrow 1} (f(x)h(x) + g(x)) = 4(0) + (-2) = -2.$$

The denominator satisfies  $\lim_{x \rightarrow 1} h(x) = 0$ .

#### Step 2: Determine whether a finite limit exists

The numerator tends to a nonzero number, so its magnitude stays bounded away from zero near  $x = 1$ . Since  $h(x)$  is nonzero in a deleted neighborhood,

$$|k(x)| = \left| \frac{f(x)h(x) + g(x)}{h(x)} \right| \rightarrow \infty.$$

There is no finite limit.

#### Step 3: Examine the sign

The sign depends on the denominator. For example, choose  $f(x) = 4$  and  $g(x) = -2$ , so that  $k(x) = 4 - 2/h(x)$ .

| Denominator $h(x)$ | Behavior of $k(x)$            |
|--------------------|-------------------------------|
| $(x - 1)^2$        | $\rightarrow -\infty$         |
| $-(x - 1)^2$       | $\rightarrow +\infty$         |
| $x - 1$            | opposite one-sided infinities |

All three choices satisfy the given limits. Therefore,

A signed infinite limit is not determined.

Division by a zero limiting value is not a valid use of the quotient law.

**Question 4**

Let

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

- (a) Find  $\lim_{x \rightarrow 3} f(x)$  using limit properties.
- (b) Determine the value of  $k$  such that  $f(x)$  is continuous at  $x = 3$ .

**Question 4 - Solution**

We are given:

$$f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

**(a) Find the limit:**

First, factor the numerator:

$$x^2 - 9 = (x - 3)(x + 3)$$

So for  $x \neq 3$ , we can simplify:

$$f(x) = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$$

Thus,

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (x + 3) = 6$$

**Answer:**  $\lim_{x \rightarrow 3} f(x) = 6$

**(b) Value of  $k$  for Continuity:**

For  $f(x)$  to be continuous at  $x = 3$ , we must have:

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

From (a), the limit is 6. Therefore, set  $f(3) = k = 6$

**Answer:**  $k = 6$

**Question 5**

Let  $f(x) = \sqrt{4x + 5}$  and  $g(x) = \frac{x^2 + 2x - 3}{x - 1}$ , and assume  $\lim_{x \rightarrow 1} f(x) = 3$ .

(a) Find  $\lim_{x \rightarrow 1} g(x)$ .

(b) Use the limit properties to find  $\lim_{x \rightarrow 1} [f(x) \cdot g(x)]$ .

**Question 5 - Solution**

We are given:

$$f(x) = \sqrt{4x + 5}, \quad g(x) = \frac{x^2 + 2x - 3}{x - 1}, \quad \lim_{x \rightarrow 1} f(x) = 3$$

**(a) Find  $\lim_{x \rightarrow 1} g(x)$ :**

First, factor the numerator of  $g(x)$ :

$$x^2 + 2x - 3 = (x + 3)(x - 1)$$

$$g(x) = \frac{(x + 3)(x - 1)}{x - 1}$$

For  $x \neq 1$ , cancel common factors:

$$g(x) = x + 3$$

So,

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x + 3) = 4$$

**Answer:** 4

**(b) Find  $\lim_{x \rightarrow 1} [f(x) \cdot g(x)]$ :**

Use the limit product law:

$$\lim_{x \rightarrow 1} [f(x) \cdot g(x)] = \lim_{x \rightarrow 1} f(x) \cdot \lim_{x \rightarrow 1} g(x) = 3 \cdot 4 = 12$$

**Answer:** 12

**Question 6**

Let  $f(x)$  and  $g(x)$  be functions such that:

$$\lim_{x \rightarrow 2} f(x) = 5, \quad \lim_{x \rightarrow 2} g(x) = -3$$

Use limit laws to compute the following:

(a)  $\lim_{x \rightarrow 2} [3f(x) - 2g(x)]$

(b)  $\lim_{x \rightarrow 2} \left[ \frac{f(x)}{g(x)} \right]$

**Question 6 - Solution**

We are given:

$$\lim_{x \rightarrow 2} f(x) = 5, \quad \lim_{x \rightarrow 2} g(x) = -3$$

(a) Use limit laws for scalar multiplication and addition/subtraction:

$$\begin{aligned} \lim_{x \rightarrow 2} [3f(x) - 2g(x)] &= 3 \cdot \lim_{x \rightarrow 2} f(x) - 2 \cdot \lim_{x \rightarrow 2} g(x) \\ &= 3(5) - 2(-3) = 15 + 6 = \boxed{21} \end{aligned}$$

(b) Use the quotient limit law:

$$\lim_{x \rightarrow 2} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow 2} f(x)}{\lim_{x \rightarrow 2} g(x)} = \frac{5}{-3} = \boxed{-\frac{5}{3}}$$

**Question 7**

Given that

$$\lim_{x \rightarrow 1} f(x) = 4, \quad \lim_{x \rightarrow 1} g(x) = 0, \quad \lim_{x \rightarrow 1} h(x) = -2$$

compute the following limits using limit laws:

(a)  $\lim_{x \rightarrow 1} [f(x)^2 + 3h(x)]$

(b)  $\lim_{x \rightarrow 1} \left[ \frac{f(x) + h(x)}{g(x)^2 + 1} \right]$

**Question 7 - Solution**

We are given:

$$\lim_{x \rightarrow 1} f(x) = 4, \quad \lim_{x \rightarrow 1} g(x) = 0, \quad \lim_{x \rightarrow 1} h(x) = -2$$

(a) Use the limit laws for powers and linear combinations:

$$\lim_{x \rightarrow 1} [f(x)^2 + 3h(x)] = (\lim_{x \rightarrow 1} f(x))^2 + 3 \cdot \lim_{x \rightarrow 1} h(x) = (4)^2 + 3(-2) = 16 - 6 = \boxed{10}$$

(b) Use limit laws for addition, powers, and quotients (denominator never zero):

$$\lim_{x \rightarrow 1} \left[ \frac{f(x) + h(x)}{g(x)^2 + 1} \right] = \frac{\lim_{x \rightarrow 1} (f(x) + h(x))}{\lim_{x \rightarrow 1} (g(x)^2 + 1)} = \frac{4 + (-2)}{0^2 + 1} = \frac{2}{1} = \boxed{2}$$

**Question 8**

Assume the following limits exist:

$$\lim_{x \rightarrow -1} f(x) = 2, \quad \lim_{x \rightarrow -1} g(x) = 3$$

Evaluate the limit:

$$\lim_{x \rightarrow -1} [(f(x) - g(x))^2 + 4f(x)]$$

Use limit laws to justify each step.

**Question 8 - Solution**

We are given:

$$\lim_{x \rightarrow -1} f(x) = 2, \quad \lim_{x \rightarrow -1} g(x) = 3$$

We evaluate:

$$\lim_{x \rightarrow -1} [(f(x) - g(x))^2 + 4f(x)]$$

**Step 1: Apply limit laws to each part**

First, consider the difference:

$$\lim_{x \rightarrow -1} (f(x) - g(x)) = \lim_{x \rightarrow -1} f(x) - \lim_{x \rightarrow -1} g(x) = 2 - 3 = -1$$

Now square the result:

$$\lim_{x \rightarrow -1} (f(x) - g(x))^2 = (-1)^2 = 1$$

Next, evaluate the linear term:

$$\lim_{x \rightarrow -1} 4f(x) = 4 \cdot \lim_{x \rightarrow -1} f(x) = 4 \cdot 2 = 8$$

**Step 2: Combine results**

Using the sum rule for limits:

$$\lim_{x \rightarrow -1} [(f(x) - g(x))^2 + 4f(x)] = 1 + 8 = \boxed{9}$$

**Question 9**

Let  $\lim_{x \rightarrow 3} f(x) = 5$  and  $\lim_{x \rightarrow 3} g(x) = -2$ .

Evaluate the following limit using limit laws:

$$\lim_{x \rightarrow 3} \left( \frac{f(x)^2 + 3g(x)}{g(x) - f(x)} \right)$$

Justify each step using properties of limits.

**Question 9 - Solution**

We are given:

$$\lim_{x \rightarrow 3} f(x) = 5, \quad \lim_{x \rightarrow 3} g(x) = -2$$

We need to evaluate:

$$\lim_{x \rightarrow 3} \left( \frac{f(x)^2 + 3g(x)}{g(x) - f(x)} \right)$$

**Step 1: Use limit laws in numerator and denominator.**

*Numerator:*

$$\lim_{x \rightarrow 3} f(x)^2 = \left( \lim_{x \rightarrow 3} f(x) \right)^2 = 5^2 = 25$$

$$\lim_{x \rightarrow 3} 3g(x) = 3 \cdot \lim_{x \rightarrow 3} g(x) = 3 \cdot (-2) = -6$$

$$\text{So, } \lim_{x \rightarrow 3} (f(x)^2 + 3g(x)) = 25 - 6 = 19$$

*Denominator:*

$$\lim_{x \rightarrow 3} (g(x) - f(x)) = \lim_{x \rightarrow 3} g(x) - \lim_{x \rightarrow 3} f(x) = -2 - 5 = -7$$

**Step 2: Final Answer**

$$\lim_{x \rightarrow 3} \left( \frac{f(x)^2 + 3g(x)}{g(x) - f(x)} \right) = \frac{19}{-7} = \boxed{-\frac{19}{7}}$$

**Question 10**

Let the functions  $f(x)$ ,  $g(x)$ , and  $h(x)$  satisfy the inequalities:

$$g(x) \leq f(x) \leq h(x)$$

for all  $x$  near 2 (but not necessarily at 2), and suppose:

$$\lim_{x \rightarrow 2} g(x) = 4 \quad \text{and} \quad \lim_{x \rightarrow 2} h(x) = 4$$

Use this information to determine:

$$\lim_{x \rightarrow 2} f(x)$$

Explain the reasoning behind your conclusion.

**Question 10 - Solution**

We are told that for all  $x$  near 2:

$$g(x) \leq f(x) \leq h(x)$$

and

$$\lim_{x \rightarrow 2} g(x) = 4, \quad \lim_{x \rightarrow 2} h(x) = 4$$

This is a perfect scenario to apply the "Squeeze Theorem", which states:

If  $g(x) \leq f(x) \leq h(x)$  near  $x = a$ , and

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} f(x) = L$$

Applying this here:

$$\lim_{x \rightarrow 2} f(x) = \boxed{4}$$

**Conclusion:** By the Squeeze Theorem, since both bounding functions tend to 4 as  $x \rightarrow 2$ , so must  $f(x)$ .