

Question 1

Let the function f be defined as:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & \text{if } x < 2 \\ x + 2, & \text{if } x \geq 2 \end{cases}$$

Evaluate:

$$\lim_{x \rightarrow 2^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x)$$

State whether $\lim_{x \rightarrow 2} f(x)$ exists.

Question 1 - Solution

We analyze the left-hand and right-hand limits separately.

Left-hand limit:

For $x < 2$,

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$$

For $x \neq 2$, the $x - 2$ cancels:

$$f(x) = x + 2$$

So:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 4$$

Right-hand limit:

For $x \geq 2$, we use the second piece:

$$f(x) = x + 2 \Rightarrow \lim_{x \rightarrow 2^+} f(x) = 4$$

Conclusion:

Since both one-sided limits exist and are equal:

$$\lim_{x \rightarrow 2} f(x) = \boxed{4}$$

Question 2

Let the function f be defined as follows:

$$f(x) = \begin{cases} 3x + 1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 1^-} f(x)$
- (b) Compute $\lim_{x \rightarrow 1^+} f(x)$
- (c) Does $\lim_{x \rightarrow 1} f(x)$ exist? Justify your answer.

Question 2 - Solution

We are given:

$$f(x) = \begin{cases} 3x + 1, & x < 1 \\ x^2, & x \geq 1 \end{cases}$$

(a) **Left-hand limit:** For $x < 1$, use $f(x) = 3x + 1$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x + 1) = 3(1) + 1 = 4$$

(b) **Right-hand limit:** For $x \geq 1$, use $f(x) = x^2$:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 = 1^2 = 1$$

(c) **Two-sided limit:** Since the left and right limits are not equal:

$$\lim_{x \rightarrow 1^-} f(x) = 4 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 1 \Rightarrow \lim_{x \rightarrow 1} f(x) \text{ does not exist}$$

Final Answer:

$\lim_{x \rightarrow 1^-} f(x) = 4, \quad \lim_{x \rightarrow 1^+} f(x) = 1, \quad \lim_{x \rightarrow 1} f(x) \text{ does not exist}$
--

Question 3

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 2^-} f(x)$
- (b) Compute $\lim_{x \rightarrow 2^+} f(x)$
- (c) Is $f(x)$ continuous at $x = 2$? Explain.

Question 3 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$$

(a) **Left-hand limit:** First, simplify the expression for $x < 2$:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2 \quad \text{for } x \neq 2$$

So,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 2 + 2 = 4$$

(b) **Right-hand limit:**

$$f(x) = x + 1 \quad \text{for } x \geq 2$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x + 1) = 2 + 1 = 3$$

(c) **Continuity at $x = 2$:** Since

$$\lim_{x \rightarrow 2^-} f(x) = 4, \quad \lim_{x \rightarrow 2^+} f(x) = 3 \Rightarrow \lim_{x \rightarrow 2} f(x) \text{ does not exist}$$

And because the limit does not exist, the function is not continuous at $x = 2$.

Final Answer:

$\lim_{x \rightarrow 2^-} f(x) = 4, \quad \lim_{x \rightarrow 2^+} f(x) = 3, \quad \text{Not continuous at } x = 2$

Question 4

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} x^2 - 4, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 1^-} f(x)$
- (b) Find $\lim_{x \rightarrow 1^+} f(x)$
- (c) Does $\lim_{x \rightarrow 1} f(x)$ exist? Explain.

Question 4 - Solution

We are given:

$$f(x) = \begin{cases} x^2 - 4, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$$

(a) Left-hand limit:

For $x < 1$, we use $f(x) = x^2 - 4$:

$$\lim_{x \rightarrow 1^-} f(x) = 1^2 - 4 = -3$$

$$\boxed{\lim_{x \rightarrow 1^-} f(x) = -3}$$

(b) Right-hand limit:

For $x \geq 1$, we use $f(x) = 2x + 1$:

$$\lim_{x \rightarrow 1^+} f(x) = 2(1) + 1 = 3$$

$$\boxed{\lim_{x \rightarrow 1^+} f(x) = 3}$$

(c) Two-sided limit:

$$\lim_{x \rightarrow 1^-} f(x) = -3 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 3$$

Since the one-sided limits are not equal, the overall limit does not exist:

$$\boxed{\lim_{x \rightarrow 1} f(x) \text{ does not exist}}$$

Question 5

Consider the function:

$$f(x) = \begin{cases} \frac{1}{x-3}, & x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 3^-} f(x)$
- (b) Find $\lim_{x \rightarrow 3^+} f(x)$
- (c) Does $\lim_{x \rightarrow 3} f(x)$ exist? Justify your answer.

Question 5 - Solution

We are given:

$$f(x) = \begin{cases} \frac{1}{x-3}, & x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

(a) Left-hand limit:

As $x \rightarrow 3^-$, we are approaching 3 from values less than 3. The expression is:

$$f(x) = \frac{1}{x-3}$$

As $x \rightarrow 3^-$, $x-3 \rightarrow 0^- \Rightarrow \frac{1}{x-3} \rightarrow -\infty$

$$\boxed{\lim_{x \rightarrow 3^-} f(x) = -\infty}$$

(b) Right-hand limit:

As $x \rightarrow 3^+$, the function becomes:

$$f(x) = \sqrt{x-3}$$

As $x \rightarrow 3^+$, $x-3 \rightarrow 0^+ \Rightarrow \sqrt{x-3} \rightarrow 0$

$$\boxed{\lim_{x \rightarrow 3^+} f(x) = 0}$$

(c) Does the limit exist?

Since

$$\lim_{x \rightarrow 3^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 3^+} f(x) = 0$$

the two one-sided limits are not equal (and one diverges), so the limit $\lim_{x \rightarrow 3} f(x)$ does not exist.

$$\boxed{\text{Limit does not exist at } x = 3}$$

Question 6

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x}, & x < 0 \\ \frac{1}{x+1}, & x \geq 0 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 0^-} f(x)$
- (b) Compute $\lim_{x \rightarrow 0^+} f(x)$
- (c) Does $\lim_{x \rightarrow 0} f(x)$ exist? Explain.

Question 6 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x}, & x < 0 \\ \frac{1}{x+1}, & x \geq 0 \end{cases}$$

(a) Left-hand limit:

For $x < 0$, use:

$$\lim_{x \rightarrow 0^-} \frac{\sin(\pi x)}{x}$$

Using the identity $\lim_{x \rightarrow 0} \frac{\sin(\pi x)}{x} = \pi$ (from standard limit $\lim_{u \rightarrow 0} \frac{\sin(u)}{u} = 1$):

$$\lim_{x \rightarrow 0^-} \frac{\sin(\pi x)}{x} = \pi$$

$$\lim_{x \rightarrow 0^-} f(x) = \pi$$

(b) Right-hand limit:

For $x \geq 0$, use:

$$\lim_{x \rightarrow 0^+} \frac{1}{x+1} = \frac{1}{1} = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

(c) Two-sided limit:

Since:

$$\lim_{x \rightarrow 0^-} f(x) = \pi \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = 1$$

These are not equal, so the limit does not exist:

$$\lim_{x \rightarrow 0} f(x) \text{ does not exist}$$

Question 7

Consider the piecewise-defined function:

$$f(x) = \begin{cases} 3x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$$

- (a) Compute $\lim_{x \rightarrow 2^-} f(x)$.
- (b) Compute $\lim_{x \rightarrow 2^+} f(x)$.
- (c) Is $\lim_{x \rightarrow 2} f(x)$ defined? Explain.
- (d) What is the value of $f(2)$?

Question 7 - Solution

We are given:

$$f(x) = \begin{cases} 3x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$$

(a) Left-hand limit as $x \rightarrow 2^-$:

Use the definition for $x < 2$:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (3x + 1) = 3(2) + 1 = 7$$

$$\boxed{\lim_{x \rightarrow 2^-} f(x) = 7}$$

(b) Right-hand limit as $x \rightarrow 2^+$:

Use the definition for $x \geq 2$:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 - 4) = 2^2 - 4 = 0$$

$$\boxed{\lim_{x \rightarrow 2^+} f(x) = 0}$$

(c) Two-sided limit:

$$\lim_{x \rightarrow 2^-} f(x) = 7, \quad \lim_{x \rightarrow 2^+} f(x) = 0$$

Since the one-sided limits are not equal:

$$\boxed{\lim_{x \rightarrow 2} f(x) \text{ does not exist}}$$

(d) Value of the function at 2:

Since $x = 2 \geq 2$, use the second case:

$$f(2) = 2^2 - 4 = 0$$

$$\boxed{f(2) = 0}$$

Question 8

Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x < 2 \\ 3x - 4, & \text{if } x \geq 2 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 2^-} f(x)$.
- (b) Find $\lim_{x \rightarrow 2^+} f(x)$.
- (c) Determine whether $\lim_{x \rightarrow 2} f(x)$ exists.
- (d) Is f continuous at $x = 2$? Justify your answer.

Question 8 - Solution

We are given:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x < 2 \\ 3x-4, & \text{if } x \geq 2 \end{cases}$$

(a) Left-hand limit:

Note: $x^2 - 4 = (x - 2)(x + 2)$, so for $x \neq 2$,

$$\frac{x^2 - 4}{x - 2} = x + 2$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 4$$

$\lim_{x \rightarrow 2^-} f(x) = 4$

(b) Right-hand limit:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3x - 4) = 6 - 4 = 2$$

$\lim_{x \rightarrow 2^+} f(x) = 2$

(c) Two-sided limit:

The left-hand and right-hand limits are not equal:

$$\lim_{x \rightarrow 2^-} f(x) = 4 \neq 2 = \lim_{x \rightarrow 2^+} f(x)$$

$\lim_{x \rightarrow 2} f(x) \text{ does not exist}$

(d) Continuity at $x = 2$:

$\lim_{x \rightarrow 2} f(x)$ does not exist, $f(2) = 3(2) - 4 = 6 - 4 = 2$ (defined)

But since the limit does not exist, f is not continuous at $x = 2$.

$f \text{ is not continuous at } x = 2$

Question 9

Consider the piecewise-defined function:

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$$

- (a) Evaluate $\lim_{x \rightarrow 0^-} f(x)$.
- (b) Evaluate $\lim_{x \rightarrow 0^+} f(x)$.
- (c) Does $\lim_{x \rightarrow 0} f(x)$ exist? Explain.
- (d) Find $f(0)$, if it exists.

Question 9 - Solution

We are given:

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$$

(a) Left-hand limit as $x \rightarrow 0^-$:

Since $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$, and this applies as $x \rightarrow 0^-$ as well:

$$\lim_{x \rightarrow 0^-} f(x) = 1$$

(b) Right-hand limit as $x \rightarrow 0^+$:

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (2x + 1) = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

(c) Two-sided limit:

Since both one-sided limits are equal:

$$\lim_{x \rightarrow 0} f(x) = 1$$

(d) Value at the point:

Use the definition for $x \geq 0$:

$$f(0) = 2(0) + 1 = 1$$

$$f(0) = 1$$

Question 10

Consider the piecewise-defined function:

$$f(x) = \begin{cases} x^2 - 3x + 2, & \text{if } x < 1 \\ \sqrt{x+3} - 2, & \text{if } x \geq 1 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 1^-} f(x)$.
- (b) Find $\lim_{x \rightarrow 1^+} f(x)$.
- (c) Determine whether $\lim_{x \rightarrow 1} f(x)$ exists.
- (d) Is f continuous at $x = 1$? Justify your answer.

Question 10 - Solution

We are given:

$$f(x) = \begin{cases} x^2 - 3x + 2, & \text{if } x < 1 \\ \sqrt{x+3} - 2, & \text{if } x \geq 1 \end{cases}$$

(a) Left-hand limit:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 - 3x + 2) = 1^2 - 3(1) + 2 = 0$$

$$\boxed{\lim_{x \rightarrow 1^-} f(x) = 0}$$

(b) Right-hand limit:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (\sqrt{x+3} - 2) = \sqrt{4} - 2 = 2 - 2 = 0$$

$$\boxed{\lim_{x \rightarrow 1^+} f(x) = 0}$$

(c) Two-sided limit:

Since both one-sided limits exist and are equal:

$$\boxed{\lim_{x \rightarrow 1} f(x) = 0}$$

(d) Continuity at $x = 1$:

We already have:

$$\lim_{x \rightarrow 1} f(x) = 0$$

Now evaluate the function at $x = 1$:

$$f(1) = \sqrt{1+3} - 2 = \sqrt{4} - 2 = 2 - 2 = 0$$

Since: $f(1)$ is defined , $\lim_{x \rightarrow 1} f(x) = f(1)$

We conclude:

$$\boxed{f \text{ is continuous at } x = 1}$$