

Question 1

Evaluate the following limit, and explain your reasoning clearly:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x} - 1}$$

Question 1 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x} - 1}$$

This limit gives an indeterminate form $\frac{0}{0}$, so we must simplify.
Start by factoring the numerator:

$$x^2 - 1 = (x - 1)(x + 1)$$

Now rewrite the expression:

$$\frac{(x - 1)(x + 1)}{\sqrt{x} - 1}$$

To simplify, multiply numerator and denominator by the ****conjugate**** of the denominator:

$$\frac{(x - 1)(x + 1)}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1}$$

This gives:

$$\frac{(x - 1)(x + 1)(\sqrt{x} + 1)}{x - 1}$$

Now cancel $x - 1$ (which is nonzero near $x = 1$):

$$(x + 1)(\sqrt{x} + 1)$$

Now substitute $x = 1$:

$$(1 + 1)(\sqrt{1} + 1) = 2 \cdot 2 = \boxed{4}$$

Question 2

Let the function be defined as:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Determine whether $\lim_{x \rightarrow 2} f(x)$ exists. If it does, find its value. Justify your answer.

Question 2 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 2} f(x)$$

where

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Step 1: Evaluate the Left-Hand Limit

For $x < 2$,

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$$

As long as $x \neq 2$, we can cancel $x - 2$:

$$f(x) = x + 2$$

So:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x + 2) = 4$$

Step 2: Evaluate the Right-Hand Limit

For $x \geq 2$, $f(x) = 3x - 2$, so:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3x - 2) = 3(2) - 2 = 6 - 2 = 4$$

Step 3: Compare One-Sided Limits

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 4 \Rightarrow \lim_{x \rightarrow 2} f(x) = \boxed{4}$$

Conclusion: The limit exists and is equal to $\boxed{4}$.

Question 3

Evaluate the limit, if it exists:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-x}}{x}$$

Explain your reasoning clearly.

Question 3 - Solution

We are asked to evaluate:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+3x} - \sqrt{1-x}}{x}$$

Direct substitution gives an indeterminate form $\frac{0}{0}$, so we simplify.

Multiply the numerator and denominator by the conjugate of the numerator:

$$\frac{\sqrt{1+3x} - \sqrt{1-x}}{x} \cdot \frac{\sqrt{1+3x} + \sqrt{1-x}}{\sqrt{1+3x} + \sqrt{1-x}}$$

This yields:

$$\frac{(1+3x) - (1-x)}{x(\sqrt{1+3x} + \sqrt{1-x})} = \frac{4x}{x(\sqrt{1+3x} + \sqrt{1-x})}$$

Cancel x :

$$\frac{4}{\sqrt{1+3x} + \sqrt{1-x}}$$

Now take the limit as $x \rightarrow 0$:

$$\frac{4}{\sqrt{1} + \sqrt{1}} = \frac{4}{2} = \boxed{2}$$

Question 4

Evaluate the following limit:

$$\lim_{x \rightarrow 5} \frac{x^2 - 25}{\sqrt{x+1} - \sqrt{6}}$$

Provide a clear justification for each step.

Question 4 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 5} \frac{x^2 - 25}{\sqrt{x+1} - \sqrt{6}}$$

First, try direct substitution:

$$\frac{5^2 - 25}{\sqrt{5+1} - \sqrt{6}} = \frac{25 - 25}{\sqrt{6} - \sqrt{6}} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.

Factor the numerator:

$$x^2 - 25 = (x - 5)(x + 5)$$

So the limit becomes:

$$\lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)}{\sqrt{x+1} - \sqrt{6}}$$

The denominator contains square roots, so we multiply by the conjugate:

$$\frac{(x - 5)(x + 5)}{\sqrt{x+1} - \sqrt{6}} \cdot \frac{\sqrt{x+1} + \sqrt{6}}{\sqrt{x+1} + \sqrt{6}}$$

This does not change the value of the expression because we are multiplying by 1.

Now simplify the denominator using the difference of squares formula:

$$(a - b)(a + b) = a^2 - b^2$$

Here,

$$a = \sqrt{x+1} \quad \text{and} \quad b = \sqrt{6}$$

Therefore,

$$(\sqrt{x+1} - \sqrt{6})(\sqrt{x+1} + \sqrt{6}) = (\sqrt{x+1})^2 - (\sqrt{6})^2$$

$$= x + 1 - 6$$

$$= x - 5$$

So the expression becomes:

$$\lim_{x \rightarrow 5} \frac{(x - 5)(x + 5)(\sqrt{x+1} + \sqrt{6})}{x - 5}$$

Since we are evaluating the limit as x approaches 5, we may cancel the common factor $x - 5$ for $x \neq 5$:

$$\lim_{x \rightarrow 5} (x + 5)(\sqrt{x+1} + \sqrt{6})$$

Now substitute $x = 5$:

$$(5 + 5)(\sqrt{5+1} + \sqrt{6})$$

$$= 10(\sqrt{6} + \sqrt{6})$$

$$= 10(2\sqrt{6})$$

$$= 20\sqrt{6}$$

Therefore,

$$\boxed{20\sqrt{6}}$$

Question 5

Evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}$$

Give a full explanation of your steps and reasoning.

Question 5 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)}$$

First, try direct substitution:

$$\frac{\sin(3 \cdot 0)}{\tan(5 \cdot 0)} = \frac{\sin(0)}{\tan(0)} = \frac{0}{0}$$

This is an indeterminate form, so we need to rewrite the expression before evaluating the limit.

We will use the standard trigonometric limits:

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$$

and

$$\lim_{u \rightarrow 0} \frac{\tan u}{u} = 1$$

Since $3x \rightarrow 0$ and $5x \rightarrow 0$ as $x \rightarrow 0$, we can apply these limits to $\sin(3x)$ and $\tan(5x)$.

Start with the original expression:

$$\frac{\sin(3x)}{\tan(5x)}$$

We want to create the forms

$$\frac{\sin(3x)}{3x} \quad \text{and} \quad \frac{\tan(5x)}{5x}$$

To do this, multiply and divide by the needed factors:

$$\frac{\sin(3x)}{\tan(5x)} = \frac{\sin(3x)}{3x} \cdot \frac{5x}{\tan(5x)} \cdot \frac{3x}{5x}$$

This rewriting is valid because the factors we introduced balance each other out.

Now take the limit of each factor:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} = 1$$

Also,

$$\lim_{x \rightarrow 0} \frac{5x}{\tan(5x)} = 1$$

This is because

$$\lim_{x \rightarrow 0} \frac{\tan(5x)}{5x} = 1$$

so its reciprocal also approaches 1.

Finally,

$$\frac{3x}{5x} = \frac{3}{5}$$

for $x \neq 0$. Since limits only depend on values near 0, not necessarily at 0, this cancellation is allowed.

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(3x)}{\tan(5x)} &= 1 \cdot 1 \cdot \frac{3}{5} \\ &= \boxed{\frac{3}{5}} \end{aligned}$$

Question 6

Evaluate the limit:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8}$$

Provide detailed steps and justification for each transformation you make.

Question 6 - Solution

We want to evaluate:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8}$$

First, try direct substitution:

$$\frac{2^2 - 4}{2^3 - 8} = \frac{4 - 4}{8 - 8} = \frac{0}{0}$$

This is an indeterminate form, so we need to simplify the expression before evaluating the limit.

Now factor the numerator:

$$x^2 - 4 = (x - 2)(x + 2)$$

This is a difference of squares because:

$$x^2 - 4 = x^2 - 2^2$$

Next, factor the denominator:

$$x^3 - 8 = x^3 - 2^3$$

This is a difference of cubes. Recall the formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Here,

$$a = x \quad \text{and} \quad b = 2$$

So:

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

Now rewrite the original limit using these factorizations:

$$\lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 2)(x^2 + 2x + 4)}$$

Both the numerator and denominator contain the factor $x - 2$. Since we are finding the limit as x approaches 2, we only need to consider values of x near 2, but not equal to 2. Therefore, for $x \neq 2$, we may cancel the common factor $x - 2$:

$$\lim_{x \rightarrow 2} \frac{x + 2}{x^2 + 2x + 4}$$

Now the expression is no longer indeterminate, so we can substitute $x = 2$:

$$\begin{aligned} & \frac{2 + 2}{2^2 + 2(2) + 4} \\ &= \frac{4}{4 + 4 + 4} \\ &= \frac{4}{12} \\ &= \frac{1}{3} \end{aligned}$$

Therefore,

$$\boxed{\frac{1}{3}}$$

Question 7

Let

$$f(x) = \begin{cases} \frac{\sin(5x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Determine the value of k that makes $f(x)$ continuous at $x = 0$.

Question 7 - Solution

We are given a piecewise function:

$$f(x) = \begin{cases} \frac{\sin(5x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

To make $f(x)$ continuous at $x = 0$, we must have:

$$\lim_{x \rightarrow 0} f(x) = f(0) = k$$

We now compute:

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{x}$$

We use the identity:

$$\lim_{x \rightarrow 0} \frac{\sin(ax)}{x} = a$$

So:

$$\lim_{x \rightarrow 0} \frac{\sin(5x)}{x} = 5$$

Thus, for continuity at $x = 0$, we must have:

$$k = \boxed{5}$$

Conclusion: The function is continuous at $x = 0$ if $\boxed{k = 5}$.

Question 8

Evaluate the following limit:

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)}$$

Question 8 - Solution

We are asked to compute:

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)}$$

First, substitute $x = 0$ into the numerator:

$$e^{2(0)} - 1 = e^0 - 1 = 1 - 1 = 0.$$

Now substitute $x = 0$ into the denominator:

$$\sin(3 \cdot 0) = \sin(0) = 0.$$

So the expression has the indeterminate form

$$\frac{0}{0}.$$

Since both the numerator and denominator approach 0, we can use L'Hopital's Rule.

Differentiate the numerator:

$$\frac{d}{dx} (e^{2x} - 1) = 2e^{2x}.$$

Differentiate the denominator:

$$\frac{d}{dx} (\sin(3x)) = 3 \cos(3x).$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)} = \lim_{x \rightarrow 0} \frac{2e^{2x}}{3 \cos(3x)}.$$

Now substitute $x = 0$:

$$\frac{2e^{2(0)}}{3 \cos(3 \cdot 0)} = \frac{2e^0}{3 \cos(0)}.$$

Since

$$e^0 = 1$$

and

$$\cos(0) = 1,$$

we get

$$\frac{2e^0}{3 \cos(0)} = \frac{2(1)}{3(1)} = \frac{2}{3}.$$

Thus,

$$\boxed{\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\sin(3x)} = \frac{2}{3}}$$

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

Question 9 - Solution

We are given:

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

First, observe that:

$$x \rightarrow 4 \Rightarrow \text{numerator } x - 4 \rightarrow 0, \quad \text{denominator } \sqrt{x} - 2 \rightarrow 0$$

So, this is an indeterminate form $\frac{0}{0}$. We rationalize the denominator to resolve it.
Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{x - 4}{\sqrt{x} - 2} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{(x - 4)(\sqrt{x} + 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \frac{(x - 4)(\sqrt{x} + 2)}{x - 4}$$

Now cancel the common factor $x - 4$, assuming $x \neq 4$:

$$= \sqrt{x} + 2$$

Now take the limit:

$$\lim_{x \rightarrow 4} \sqrt{x} + 2 = \sqrt{4} + 2 = 2 + 2 = \boxed{4}$$

Question 10

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

Question 10 - Solution

We are given:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

We want to simplify the expression to match a known limit. Recall the standard limit:

$$\lim_{u \rightarrow 0} \frac{\sin(u)}{u} = 1$$

Let $u = 3x$. As $x \rightarrow 0$, we also have $u \rightarrow 0$.

So:

$$\frac{\sin(3x)}{x} = \frac{\sin(3x)}{3x} \cdot 3$$

Now apply the limit:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{3x} \cdot 3 \right) = \left(\lim_{u \rightarrow 0} \frac{\sin(u)}{u} \right) \cdot 3 = 1 \cdot 3 = \boxed{3}$$