

Question 1

Let

$$f(x) = 4x - 1$$

Use the ε - δ definition of the limit to prove that:

$$\lim_{x \rightarrow 2} f(x) = 7$$

Question 1 - Solution

We are given:

$$f(x) = 4x - 1, \quad \lim_{x \rightarrow 2} f(x) = 7$$

We will prove this using the ε - δ definition:

Goal: For every $\varepsilon > 0$, we must find $\delta > 0$ such that:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 7| < \varepsilon$$

Step 1: Write out $|f(x) - 7|$:

$$|f(x) - 7| = |4x - 1 - 7| = |4x - 8| = 4|x - 2|$$

Step 2: Make $4|x - 2| < \varepsilon$:

We can achieve this by choosing:

$$|x - 2| < \frac{\varepsilon}{4}$$

Step 3: Let $\delta = \frac{\varepsilon}{4}$.

Then, for all x such that $0 < |x - 2| < \delta$, we have:

$$|f(x) - 7| = 4|x - 2| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

Conclusion:

Therefore, $\lim_{x \rightarrow 2} f(x) = 7$ by the ε - δ definition.

Question 2

Let

$$f(x) = x^2 + 3x$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 1} f(x) = 4$$

Question 2 - Solution

We are given:

$$f(x) = x^2 + 3x, \quad \lim_{x \rightarrow 1} f(x) = 4$$

We want to prove using the ε - δ definition:

Goal: For every $\varepsilon > 0$, we must find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

Step 1: Write the expression $|f(x) - 4|$:

$$|f(x) - 4| = |x^2 + 3x - 4|$$

Factor the expression:

$$x^2 + 3x - 4 = (x - 1)(x + 4) \Rightarrow |f(x) - 4| = |x - 1||x + 4|$$

We want:

$$|x - 1||x + 4| < \varepsilon$$

Step 2: Estimate $|x + 4|$:

Assume $|x - 1| < 1 \Rightarrow x \in (0, 2) \Rightarrow x + 4 \in (4, 6) \Rightarrow |x + 4| < 6$

Then:

$$|f(x) - 4| = |x - 1||x + 4| < 6|x - 1|$$

To make $|f(x) - 4| < \varepsilon$, it is enough to make:

$$6|x - 1| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{6}$$

Step 3: Let $\delta = \min \left\{ 1, \frac{\varepsilon}{6} \right\}$

Then if $0 < |x - 1| < \delta$, we have:

$$|f(x) - 4| < \varepsilon$$

Conclusion:

Therefore, $\lim_{x \rightarrow 1} (x^2 + 3x) = 4$ by the ε - δ definition.

Question 3

Let

$$f(x) = 5x - 1$$

Prove, using the ε - δ definition of a limit, that:

$$\lim_{x \rightarrow 2} f(x) = 9$$

Question 3 - Solution

We are given:

$$f(x) = 5x - 1, \quad \lim_{x \rightarrow 2} f(x) = 9$$

We will use the ε - δ definition of a limit:

Goal: For every $\varepsilon > 0$, find $\delta > 0$ such that:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 9| < \varepsilon$$

Step 1: Express $|f(x) - 9|$:

$$|f(x) - 9| = |5x - 1 - 9| = |5x - 10| = 5|x - 2|$$

We want:

$$5|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ :

Let:

$$\delta = \frac{\varepsilon}{5}$$

Then if $0 < |x - 2| < \delta$, we have:

$$|f(x) - 9| = 5|x - 2| < 5\delta = \varepsilon$$

Conclusion:

Therefore, $\lim_{x \rightarrow 2} (5x - 1) = 9$ by the ε - δ definition.

Question 4

Let

$$f(x) = \frac{1}{x}$$

Prove, using the ε - δ definition of a limit, that:

$$\lim_{x \rightarrow 1} \frac{1}{x} = 1$$

Question 4 - Solution

We are given:

$$f(x) = \frac{1}{x}, \quad \lim_{x \rightarrow 1} f(x) = 1$$

We will prove this using the ε - δ definition of the limit.

Goal: For every $\varepsilon > 0$, find $\delta > 0$ such that:

$$0 < |x - 1| < \delta \quad \Rightarrow \quad \left| \frac{1}{x} - 1 \right| < \varepsilon$$

Step 1: Manipulate the expression

We begin by simplifying:

$$\left| \frac{1}{x} - 1 \right| = \left| \frac{1 - x}{x} \right| = \frac{|x - 1|}{|x|}$$

So we want:

$$\frac{|x - 1|}{|x|} < \varepsilon$$

Step 2: Bound $|x|$ away from zero

Choose $\delta \leq \frac{1}{2}$, so $|x - 1| < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$

This gives us a lower bound: $|x| > \frac{1}{2}$

So:

$$\frac{|x - 1|}{|x|} < \frac{|x - 1|}{1/2} = 2|x - 1|$$

Step 3: Ensure $2|x - 1| < \varepsilon$

To make this less than ε , we require:

$$2|x - 1| < \varepsilon \quad \Rightarrow \quad |x - 1| < \frac{\varepsilon}{2}$$

Step 4: Choose δ

Let:

$$\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$$

Then whenever $0 < |x - 1| < \delta$, we get:

$$\left| \frac{1}{x} - 1 \right| = \frac{|x - 1|}{|x|} < 2|x - 1| < 2\delta \leq \varepsilon$$

Conclusion:

Thus, by the ε - δ definition of a limit,

$$\boxed{\lim_{x \rightarrow 1} \frac{1}{x} = 1}$$

Question 5

Let

$$f(x) = 5x - 3$$

Prove, using the ε - δ definition of a limit, that:

$$\lim_{x \rightarrow 2} (5x - 3) = 7$$

Question 5 - Solution

We are given:

$$f(x) = 5x - 3$$

We want to prove:

$$\lim_{x \rightarrow 2} (5x - 3) = 7$$

using the formal ε - δ definition of the limit.

Definition: For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |(5x - 3) - 7| < \varepsilon$$

Step 1: Simplify the expression

$$|(5x - 3) - 7| = |5x - 10| = 5|x - 2|$$

So we want:

$$5|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ

Let:

$$\delta = \frac{\varepsilon}{5}$$

Step 3: Conclusion

Then whenever $0 < |x - 2| < \delta$, we have:

$$|(5x - 3) - 7| = 5|x - 2| < 5\delta = \varepsilon$$

Therefore, by the ε - δ definition,

$$\boxed{\lim_{x \rightarrow 2} (5x - 3) = 7}$$

Question 6

Let

$$f(x) = \frac{1}{2}x + 4$$

Prove using the ε - δ definition of a limit that:

$$\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 4 \right) = 3$$

Question 6 - Solution

We are given:

$$f(x) = \frac{1}{2}x + 4$$

and want to prove:

$$\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 4 \right) = 3$$

using the formal ε - δ definition of a limit.

Definition: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x + 2| < \delta \quad \Rightarrow \quad \left| \left(\frac{1}{2}x + 4 \right) - 3 \right| < \varepsilon$$

Step 1: Simplify the expression

$$\left| \left(\frac{1}{2}x + 4 \right) - 3 \right| = \left| \frac{1}{2}x + 1 \right| = \left| \frac{1}{2}(x + 2) \right| = \frac{1}{2}|x + 2|$$

So we want:

$$\frac{1}{2}|x + 2| < \varepsilon \quad \Rightarrow \quad |x + 2| < 2\varepsilon$$

Step 2: Choose δ

Let:

$$\delta = 2\varepsilon$$

Step 3: Conclusion

Then whenever $0 < |x + 2| < \delta$, we have:

$$\left| \left(\frac{1}{2}x + 4 \right) - 3 \right| = \frac{1}{2}|x + 2| < \frac{1}{2} \cdot \delta = \frac{1}{2}(2\varepsilon) = \varepsilon$$

Therefore, by the ε - δ definition,

$$\boxed{\lim_{x \rightarrow -2} \left(\frac{1}{2}x + 4 \right) = 3}$$

Question 7

Let

$$f(x) = 3x - 1$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 2} f(x) = 5$$

Question 7 - Solution

We are given the function:

$$f(x) = 3x - 1$$

and we are asked to prove:

$$\lim_{x \rightarrow 2} (3x - 1) = 5$$

using the ε - δ definition.

Step 1: Recall the definition.

We must show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |(3x - 1) - 5| < \varepsilon$$

Step 2: Simplify the expression inside the absolute value.

$$|(3x - 1) - 5| = |3x - 6| = 3|x - 2|$$

So we require:

$$3|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \frac{\varepsilon}{3}$$

Step 3: Choose δ .

Let

$$\delta = \frac{\varepsilon}{3}$$

Step 4: Verification.

If $0 < |x - 2| < \delta = \frac{\varepsilon}{3}$, then:

$$|f(x) - 5| = |3x - 6| = 3|x - 2| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

Conclusion: By the ε - δ definition, we have:

$$\boxed{\lim_{x \rightarrow 2} (3x - 1) = 5}$$

Question 8

Let

$$f(x) = x^2$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 3} f(x) = 9$$

Question 8 - Solution

We are given:

$$f(x) = x^2, \quad \lim_{x \rightarrow 3} x^2 = 9$$

We want to prove that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon$$

Step 1: Manipulate the expression.

$$|x^2 - 9| = |x - 3||x + 3|$$

We want to bound this expression. Assume $|x - 3| < 1$. Then:

$$2 < x < 4 \Rightarrow 5 < x + 3 < 7 \Rightarrow |x + 3| < 7$$

So:

$$|x^2 - 9| = |x - 3||x + 3| < |x - 3| \cdot 7$$

We want:

$$|x - 3| \cdot 7 < \varepsilon \Rightarrow |x - 3| < \frac{\varepsilon}{7}$$

Step 2: Choose δ .

Let:

$$\delta = \min\left(1, \frac{\varepsilon}{7}\right)$$

Step 3: Verify.

If $0 < |x - 3| < \delta$, then:

$$|x^2 - 9| = |x - 3||x + 3| < \delta \cdot 7 \leq \frac{\varepsilon}{7} \cdot 7 = \varepsilon$$

Conclusion: By the ε - δ definition of the limit:

$$\lim_{x \rightarrow 3} x^2 = 9$$

Question 9

Let

$$f(x) = 5x - 1$$

Use the ε - δ definition of a limit to prove that:

$$\lim_{x \rightarrow 2} f(x) = 9$$

Question 9 - Solution

We are given:

$$f(x) = 5x - 1, \quad \lim_{x \rightarrow 2} (5x - 1) = 9$$

We want to show that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta, \text{ then } |f(x) - 9| < \varepsilon$$

Step 1: Work with the expression $|f(x) - 9|$

$$|f(x) - 9| = |5x - 1 - 9| = |5x - 10| = 5|x - 2|$$

We want:

$$5|x - 2| < \varepsilon \Rightarrow |x - 2| < \frac{\varepsilon}{5}$$

Step 2: Choose δ

Let:

$$\delta = \frac{\varepsilon}{5}$$

Step 3: Conclusion

Then for all $\varepsilon > 0$, if $0 < |x - 2| < \delta$, we have:

$$|f(x) - 9| = 5|x - 2| < 5 \cdot \delta = \varepsilon$$

Therefore, by the ε - δ definition of the limit:

$$\boxed{\lim_{x \rightarrow 2} (5x - 1) = 9}$$

Question 10

Let

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Use the ε - δ definition of a limit to prove:

$$\lim_{x \rightarrow 1} f(x) = 2$$

Question 10 - Solution

We are given:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Note that:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1 \quad \text{for } x \neq 1$$

So for $x \neq 1$, $f(x) = x+1$, and:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x+1) = 2$$

We now prove this using the ε - δ definition.

We want to show:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \varepsilon$$

Since $f(x) = x+1$ for $x \neq 1$, we compute:

$$|f(x) - 2| = |x+1-2| = |x-1|$$

So:

$$|f(x) - 2| < \varepsilon \quad \text{whenever} \quad |x - 1| < \varepsilon$$

Step: Choose $\delta = \varepsilon$

Then if $0 < |x - 1| < \delta$, we get:

$$|f(x) - 2| = |x - 1| < \varepsilon$$

Conclusion: By the ε - δ definition, we have:

$$\boxed{\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2}$$