

Question 1

Let $f(x) = \sqrt{x+4}$. Find the slope of the secant line through the points where $x = 0$ and $x = 5$, and compare it with the slope of the tangent line at $x = 2$. Then, write the equation of the tangent line at $x = 2$.

Question 1 - Solution

We are given:

$$f(x) = \sqrt{x+4}$$

Step 1: Slope of secant line from $x = 0$ to $x = 5$.

$$f(0) = \sqrt{4} = 2, \quad f(5) = \sqrt{9} = 3$$

$$\text{Secant slope} = \frac{3-2}{5-0} = \frac{1}{5}$$

Step 2: Derivative of $f(x)$.

$$f'(x) = \frac{1}{2\sqrt{x+4}}$$

$$f'(2) = \frac{1}{2\sqrt{6}} \approx 0.204$$

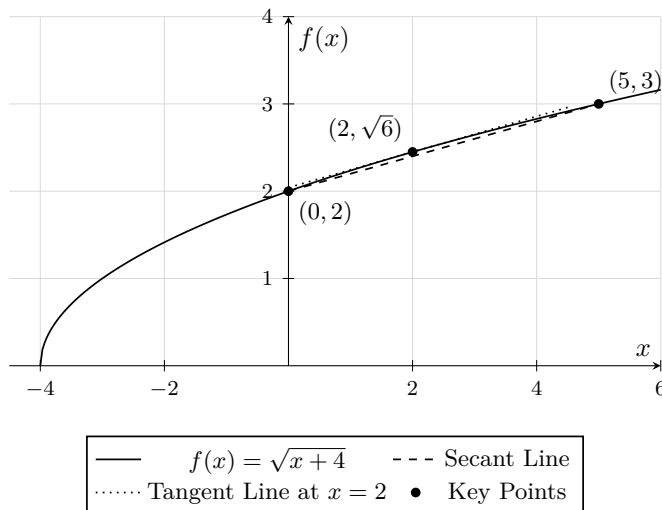
Step 3: Point at $x = 2$.

$$f(2) = \sqrt{6}$$

Step 4: Tangent line at $x = 2$:

$$y - \sqrt{6} = \frac{1}{2\sqrt{6}}(x - 2)$$

Comparison: The slope of the tangent line at $x = 2$ is slightly greater than the secant slope over the interval, showing that the graph is concave down.



Question 2

The displacement of a car traveling along a straight road is given by

$$s(t) = 10t - 0.5t^2$$

where s is in meters and t is in seconds.

- (a) Find the average velocity of the car over the interval $[2, 6]$.
- (b) Find the instantaneous velocity at $t = 2$.
- (c) Sketch the displacement function and the secant and tangent lines described above.

Question 2 - Solution

We are given:

$$s(t) = 10t - 0.5t^2$$

(a) **Average velocity on** $[2, 6]$:

$$s(2) = 10(2) - 0.5(4) = 20 - 2 = 18$$

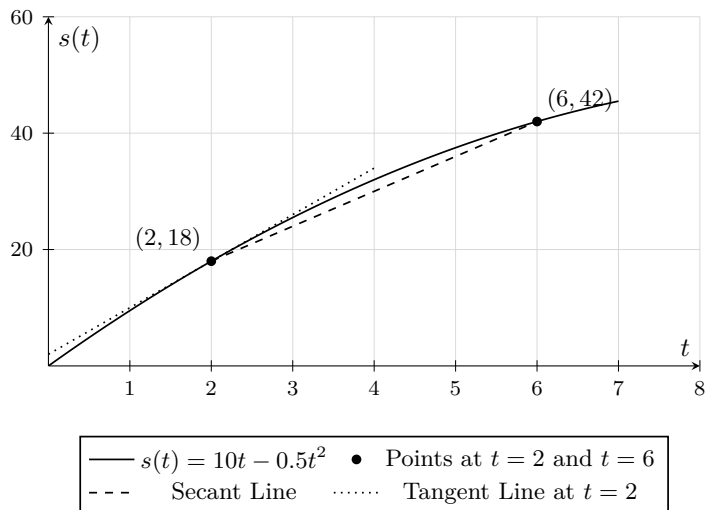
$$s(6) = 10(6) - 0.5(36) = 60 - 18 = 42$$

$$\text{Average velocity} = \frac{42 - 18}{6 - 2} = \frac{24}{4} = 6 \text{ m/s}$$

(b) **Instantaneous velocity at** $t = 2$:

$$s'(t) = 10 - t \Rightarrow s'(2) = 10 - 2 = 8 \text{ m/s}$$

(c) **Interpretation:** The average velocity between $t = 2$ and $t = 6$ is 6 m/s. At $t = 2$, the car is moving faster than average at 8 m/s. We now sketch the curve with the secant line between $t = 2$ and $t = 6$, and the tangent line at $t = 2$.



Question 3

A population of bacteria in a lab experiment is modeled by the function

$$P(t) = 200e^{0.3t},$$

where t is the time in hours since the start of the experiment and $P(t)$ is the number of bacteria.

- (a) Estimate the average rate of growth of the population from $t = 2$ to $t = 5$.
- (b) Find the instantaneous rate of growth at $t = 2$.
- (c) Sketch the population function along with the secant and tangent lines.

Question 3 - Solution

We are given:

$$P(t) = 200e^{0.3t}$$

(a) Average rate of change on $[2, 5]$:

$$P(2) = 200e^{0.6} \approx 200(1.8221) = 364.42$$

$$P(5) = 200e^{1.5} \approx 200(4.4817) = 896.34$$

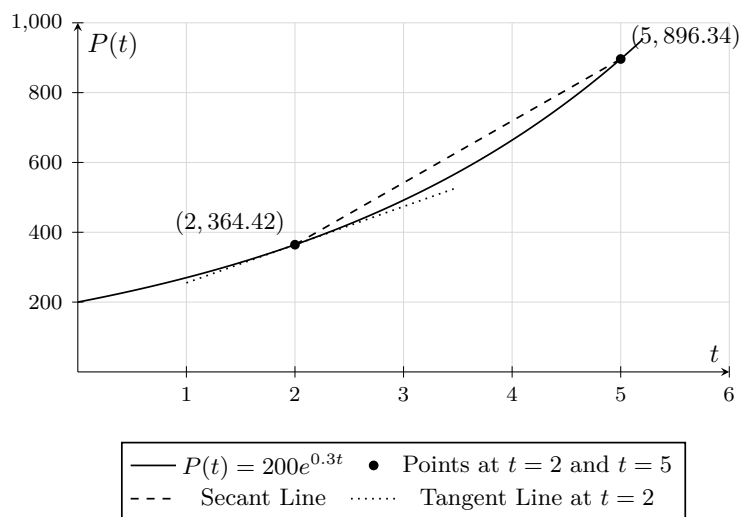
$$\text{Average rate} = \frac{896.34 - 364.42}{5 - 2} \approx \frac{531.92}{3} \approx 177.31 \text{ bacteria/hr}$$

(b) Instantaneous rate at $t = 2$:

$$P'(t) = 200 \cdot 0.3e^{0.3t} = 60e^{0.3t}$$

$$P'(2) = 60e^{0.6} \approx 60(1.8221) = 109.33 \text{ bacteria/hr}$$

(c) **Interpretation:** The population grows faster over the longer interval than it does exactly at $t = 2$, illustrating exponential acceleration.



Question 4

A ball is thrown vertically upward from the ground with an initial velocity of $v_0 = 20$ m/s. Its height at time t seconds is modeled by the function

$$h(t) = 20t - 4.9t^2.$$

- (a) Compute the average velocity over the interval $[1, 2]$.
- (b) Find the instantaneous velocity at $t = 1$.
- (c) Sketch the height function and both the secant and tangent lines involved.

Question 4 - Solution

Given:

$$h(t) = 20t - 4.9t^2$$

(a) Average velocity on $[1, 2]$:

$$h(1) = 20(1) - 4.9(1)^2 = 20 - 4.9 = 15.1$$

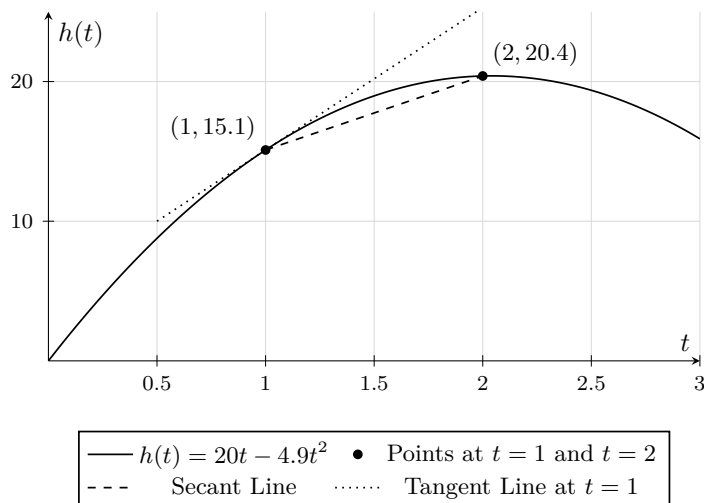
$$h(2) = 20(2) - 4.9(4) = 40 - 19.6 = 20.4$$

$$\text{Average velocity} = \frac{20.4 - 15.1}{2 - 1} = 5.3 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$h'(t) = 20 - 9.8t \Rightarrow h'(1) = 20 - 9.8 = 10.2 \text{ m/s}$$

(c) **Interpretation:** The ball's velocity at exactly $t = 1$ is greater than the average velocity over the interval, indicating it is still on the way up. Below is the sketch of the graph.



Question 5

The position of a particle moving along a straight path is given by the function

$$s(t) = t^3 - 6t^2 + 9t,$$

where $s(t)$ is in meters and t is in seconds.

- (a) Compute the average velocity of the particle on the interval $[1, 3]$.
- (b) Find the instantaneous velocity at $t = 1$.
- (c) Sketch the position function and include both the secant line over $[1, 3]$ and the tangent line at $t = 1$.

Question 5 - Solution

Given:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Average velocity on $[1, 3]$:

$$s(1) = 1^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$$

$$s(3) = 27 - 54 + 27 = 0$$

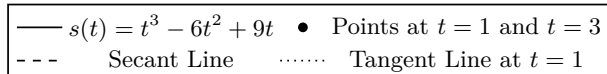
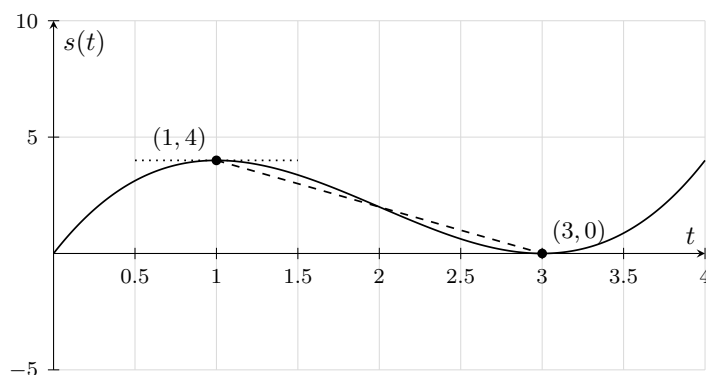
$$\text{Average velocity} = \frac{0 - 4}{3 - 1} = \frac{-4}{2} = -2 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$s'(t) = 3t^2 - 12t + 9$$

$$s'(1) = 3(1)^2 - 12(1) + 9 = 3 - 12 + 9 = 0 \text{ m/s}$$

(c) **Interpretation:** The particle is momentarily at rest at $t = 1$ (instantaneous velocity is 0), even though its average velocity between $t = 1$ and $t = 3$ is negative, indicating motion reversal.



Question 6

The height h (in meters) of a rocket launched vertically is given by the function

$$h(t) = 100 + 40t - 5t^2,$$

where t is time in seconds after launch.

- (a) Compute the average velocity of the rocket over the time interval $[2, 4]$.
- (b) Find the instantaneous velocity at $t = 2$.
- (c) Sketch the height function, the secant line over $[2, 4]$, and the tangent line at $t = 2$.

Question 6 - Solution

Given:

$$h(t) = 100 + 40t - 5t^2$$

(a) Average velocity on $[2, 4]$:

$$h(2) = 100 + 80 - 20 = 160$$

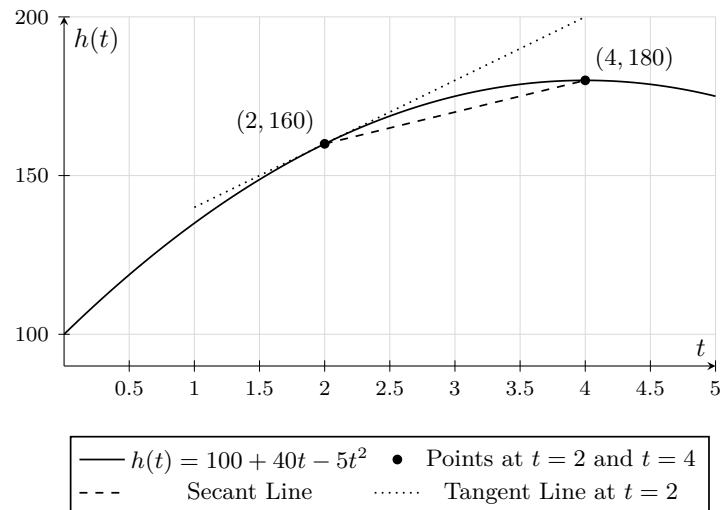
$$h(4) = 100 + 160 - 80 = 180$$

$$\text{Average velocity} = \frac{180 - 160}{4 - 2} = \frac{20}{2} = 10 \text{ m/s}$$

(b) Instantaneous velocity at $t = 2$:

$$h'(t) = 40 - 10t \Rightarrow h'(2) = 40 - 20 = 20 \text{ m/s}$$

(c) **Interpretation:** At $t = 2$, the rocket is climbing at a rate of 20 m/s, which is faster than its average velocity between $t = 2$ and $t = 4$.



Question 7

The temperature T (in degrees Celsius) of a cooling object is modeled by

$$T(t) = 25 + 60e^{-0.4t},$$

where t is time in minutes.

- (a) Compute the average rate of change of the temperature over the interval $[1, 4]$.
- (b) Find the instantaneous rate of change of the temperature at $t = 1$.
- (c) Sketch the temperature function, the secant line over $[1, 4]$, and the tangent line at $t = 1$.

Question 7 - Solution

Given:

$$T(t) = 25 + 60e^{-0.4t}$$

(a) Average rate of change on $[1, 4]$:

$$T(1) = 25 + 60e^{-0.4} \approx 25 + 60(0.6703) \approx 65.22$$

$$T(4) = 25 + 60e^{-1.6} \approx 25 + 60(0.2019) \approx 37.11$$

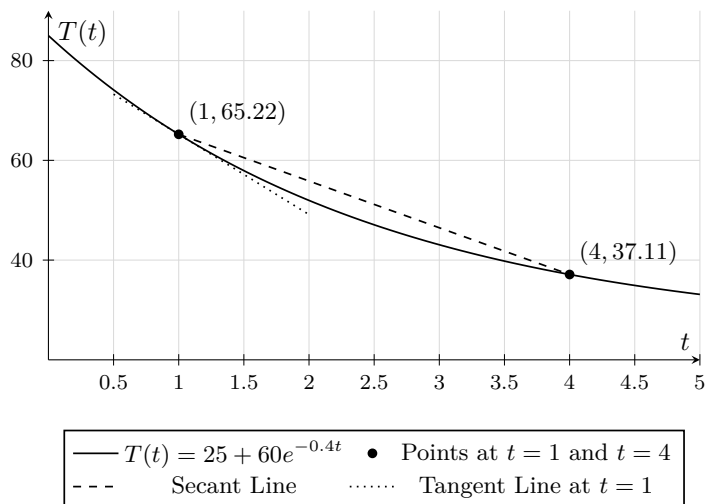
$$\text{Average rate} = \frac{37.11 - 65.22}{4 - 1} \approx \frac{-28.11}{3} \approx -9.37 \text{ }^{\circ}\text{C/min}$$

(b) Instantaneous rate of change at $t = 1$:

$$T'(t) = 60(-0.4)e^{-0.4t} = -24e^{-0.4t}$$

$$T'(1) = -24e^{-0.4} \approx -24(0.6703) \approx -16.09 \text{ }^{\circ}\text{C/min}$$

(c) **Interpretation:** At $t = 1$, the object is cooling at a faster rate than the average cooling over the interval $[1, 4]$, consistent with exponential decay.



Question 8

The population P (in thousands) of a bacteria culture after t hours is modeled by

$$P(t) = \frac{100t}{1 + 0.5t}.$$

- (a) Compute the average rate of change of the population on the interval $[1, 3]$.
- (b) Find the instantaneous rate of change at $t = 1$.
- (c) Sketch the population function, the secant line over $[1, 3]$, and the tangent line at $t = 1$.

Question 8 - Solution

Given:

$$P(t) = \frac{100t}{1 + 0.5t}$$

(a) Average rate of change on $[1, 3]$:

$$P(1) = \frac{100(1)}{1 + 0.5(1)} = \frac{100}{1.5} \approx 66.67$$

$$P(3) = \frac{100(3)}{1 + 0.5(3)} = \frac{300}{2.5} = 120$$

$$\text{Average rate} = \frac{120 - 66.67}{3 - 1} = \frac{53.33}{2} = 26.67 \text{ thousand/hour}$$

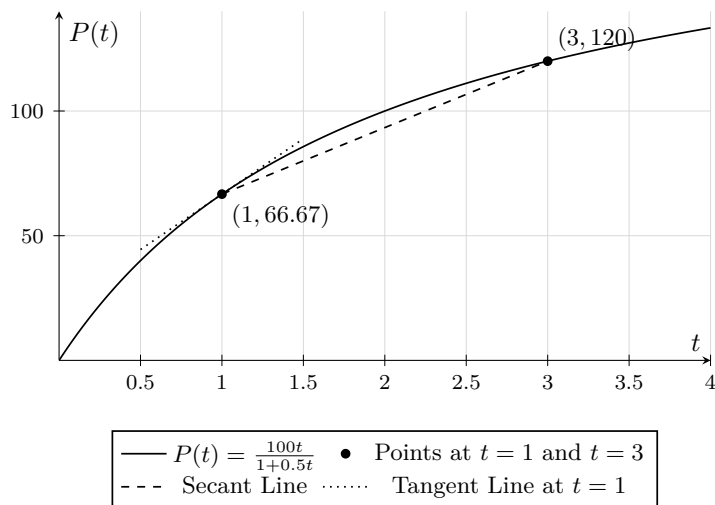
(b) Instantaneous rate of change at $t = 1$:

Use the quotient rule:

$$P'(t) = \frac{(1 + 0.5t)(100) - 100t(0.5)}{(1 + 0.5t)^2} = \frac{100 + 50t - 50t}{(1 + 0.5t)^2} = \frac{100}{(1 + 0.5t)^2}$$

$$P'(1) = \frac{100}{(1 + 0.5)^2} = \frac{100}{2.25} \approx 44.44 \text{ thousand/hour}$$

(c) **Interpretation:** The population is increasing more rapidly at $t = 1$ than the average over the interval. The tangent line shows the instantaneous trend, and the secant line reflects the average change.



Question 9

The revenue R (in dollars) generated by selling t hundred items is given by the function

$$R(t) = 200t - 3t^2.$$

- (a) Compute the average rate of change of revenue on the interval $[2, 6]$.
- (b) Find the instantaneous rate of change at $t = 2$.
- (c) Sketch the revenue function, the secant line over $[2, 6]$, and the tangent line at $t = 2$.

Question 9 - Solution

Given:

$$R(t) = 200t - 3t^2$$

(a) Average rate of change on $[2, 6]$:

$$R(2) = 200(2) - 3(2)^2 = 400 - 12 = 388$$

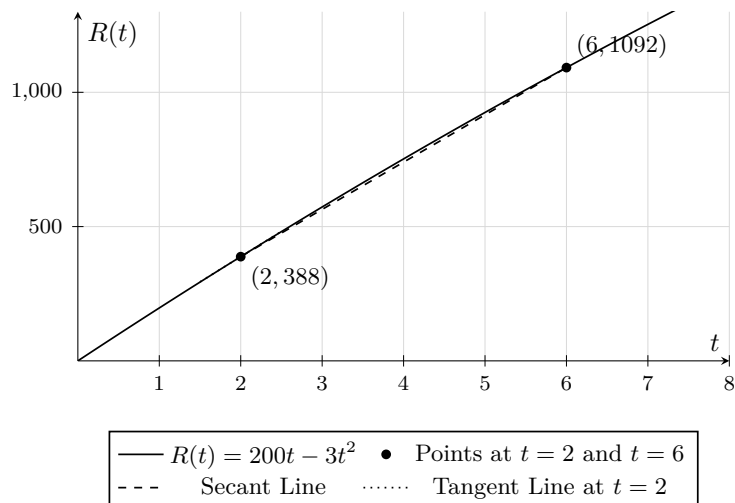
$$R(6) = 200(6) - 3(6)^2 = 1200 - 108 = 1092$$

$$\text{Average rate} = \frac{1092 - 388}{6 - 2} = \frac{704}{4} = 176 \text{ dollars per hundred items}$$

(b) Instantaneous rate of change at $t = 2$:

$$R'(t) = 200 - 6t \Rightarrow R'(2) = 200 - 12 = 188$$

(c) **Interpretation:** At $t = 2$, revenue is increasing faster than the average over the interval $[2, 6]$. The tangent line shows this steeper slope.



Question 10

The displacement $s(t)$ (in meters) of a particle moving along a straight line is given by

$$s(t) = t^3 - 6t^2 + 9t,$$

where t is time in seconds.

- (a) Find the average velocity of the particle over the interval $[1, 4]$.
- (b) Find the instantaneous velocity of the particle at $t = 1$.
- (c) Sketch the displacement function, the secant line over $[1, 4]$, and the tangent line at $t = 1$.

Question 10 - Solution

Given:

$$s(t) = t^3 - 6t^2 + 9t$$

(a) Average velocity on $[1, 4]$:

$$s(1) = 1^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$$

$$s(4) = 64 - 96 + 36 = 4$$

$$\text{Average velocity} = \frac{4 - 4}{4 - 1} = 0 \text{ m/s}$$

(b) Instantaneous velocity at $t = 1$:

$$s'(t) = 3t^2 - 12t + 9 \Rightarrow s'(1) = 3(1)^2 - 12(1) + 9 = 3 - 12 + 9 = 0 \text{ m/s}$$

(c) **Interpretation:** The average and instantaneous velocities are both zero, indicating the particle returns to its starting displacement at $t = 4$, and is momentarily at rest at $t = 1$.

