

Question 1

Solve the equation for x , showing all steps:

$$\log_3(x + 5) + \log_3(x - 2) = 2$$

- (a) Solve the logarithmic equation algebraically.
- (b) State any restrictions on the domain.
- (c) Check for extraneous solutions.

Question 1 - Solution

(a) Combine the logs using log rules:

$$\log_3(x+5) + \log_3(x-2) = \log_3[(x+5)(x-2)] = 2 \Rightarrow \log_3(x^2 + 3x - 10) = 2$$

Rewrite in exponential form:

$$x^2 + 3x - 10 = 3^2 = 9 \Rightarrow x^2 + 3x - 10 = 9 \Rightarrow x^2 + 3x - 19 = 0$$

Solve with quadratic formula:

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-19)}}{2(1)} = \frac{-3 \pm \sqrt{9 + 76}}{2} = \frac{-3 \pm \sqrt{85}}{2}$$

(b) **Domain restrictions:**

Since $\log_3(x+5)$ and $\log_3(x-2)$ are defined only if:

$$x+5 > 0 \Rightarrow x > -5, \quad x-2 > 0 \Rightarrow x > 2 \Rightarrow \text{Overall domain: } x > 2$$

(c) **Check solutions:**

$$\sqrt{85} \approx 9.22 \Rightarrow x_1 = \frac{-3 + 9.22}{2} \approx 3.11, \quad x_2 = \frac{-3 - 9.22}{2} \approx -6.11$$

Only $x_1 \approx 3.11$ satisfies $x > 2$. So:

$$\boxed{x = \frac{-3 + \sqrt{85}}{2}} \quad (\text{valid solution})$$

Question 2

A scientist is studying the pH level of a solution, which is defined by the formula:

$$\text{pH} = -\log_{10}[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions in moles per liter.

- (a) If a solution has a hydrogen ion concentration of $[H^+] = 3.2 \times 10^{-5}$, find its pH.
- (b) If another solution has a pH of 4.6, what is its hydrogen ion concentration?
- (c) Which solution is more acidic, and by how many times is it more acidic?

Question 2 - Solution

(a) **Compute pH:**

$$\begin{aligned}\text{pH} &= -\log_{10}(3.2 \times 10^{-5}) = -[\log_{10}(3.2) + \log_{10}(10^{-5})] \\ &= -[\log_{10}(3.2) - 5] \approx -[0.5051 - 5] = -(-4.4949) = \boxed{4.49}\end{aligned}$$

(b) **Compute hydrogen ion concentration from pH:**

$$\text{pH} = -\log_{10}[H^+] \Rightarrow \log_{10}[H^+] = -4.6 \Rightarrow [H^+] = 10^{-4.6} \approx \boxed{2.51 \times 10^{-5}}$$

(c) **Compare acidity:**

Recall: lower pH = more acidic.

The first solution has pH ≈ 4.49 , second has pH 4.6, so first solution is more acidic.

To find how many times more acidic:

$$\frac{[H^+]_{\text{first}}}{[H^+]_{\text{second}}} = \frac{3.2 \times 10^{-5}}{2.51 \times 10^{-5}} \approx \boxed{1.27 \text{ times more acidic}}$$

Question 3

Consider the function:

$$f(x) = \log_2(4x^2 - 12x)$$

- (a) Determine the domain of the function.
- (b) Solve the equation $f(x) = 3$.
- (c) State any restrictions or extraneous solutions.

Question 3 - Solution**(a) Domain:**

We must have:

$$4x^2 - 12x > 0 \Rightarrow 4x(x - 3) > 0$$

Using a sign chart or test intervals:

- Critical points: $x = 0$, $x = 3$ - Positive intervals: $x < 0$ or $x > 3$

$$\text{Domain: } \boxed{(-\infty, 0) \cup (3, \infty)}$$

(b) Solve $f(x) = 3$:

$$\log_2(4x^2 - 12x) = 3 \Rightarrow 4x^2 - 12x = 2^3 = 8 \Rightarrow 4x^2 - 12x - 8 = 0$$

Divide by 4:

$$x^2 - 3x - 2 = 0 \Rightarrow x = \frac{3 \pm \sqrt{(-3)^2 + 4 \cdot 2}}{2} = \frac{3 \pm \sqrt{9 + 8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

$$\boxed{x = \frac{3 \pm \sqrt{17}}{2}}$$

(c) Check for extraneous solutions:

Approximate roots:

$$\sqrt{17} \approx 4.12 \Rightarrow x \approx \frac{3 \pm 4.12}{2} \Rightarrow x_1 \approx 3.56, \quad x_2 \approx -0.56$$

From domain: $x < 0$ or $x > 3$, so both are valid.

$$\boxed{\text{No extraneous solutions}}$$

Question 4

Let $f(x) = \ln(x^2 - 6x + 10)$.

- (a) Determine the domain of $f(x)$.
- (b) Find the exact value of $f(3)$.
- (c) Solve the equation $\ln(x^2 - 6x + 10) = 0$.

Question 4 - Solution**(a) Domain:**

We require the argument of the logarithm to be positive:

$$x^2 - 6x + 10 > 0$$

Consider the quadratic:

$$x^2 - 6x + 10 = (x - 3)^2 + 1$$

Since $(x - 3)^2 \geq 0$, we know:

$$(x - 3)^2 + 1 \geq 1 > 0 \quad \text{for all real } x$$

Domain: $(-\infty, \infty)$

(b) Evaluate $f(3)$:

$$f(3) = \ln(3^2 - 6 \cdot 3 + 10) = \ln(9 - 18 + 10) = \ln(1) = \boxed{0}$$

(c) Solve $\ln(x^2 - 6x + 10) = 0$:

$$\ln(x^2 - 6x + 10) = 0 \Rightarrow x^2 - 6x + 10 = e^0 = 1 \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x - 3)^2 = 0 \Rightarrow \boxed{x = 3}$$

Question 5

Let $f(x) = \log_5(x - 1) + \log_5(x + 3)$.

- (a) Determine the domain of $f(x)$.
- (b) Simplify the expression using logarithmic properties.
- (c) Solve the equation $\log_5(x - 1) + \log_5(x + 3) = 2$.

Question 5 - Solution**(a) Domain:**

We need both logarithmic expressions to be defined:

$$x - 1 > 0 \Rightarrow x > 1, \quad x + 3 > 0 \Rightarrow x > -3$$

The more restrictive condition is $x > 1$, so:

Domain: $(1, \infty)$

(b) Use the product rule for logarithms:

$$f(x) = \log_5[(x-1)(x+3)] = \log_5(x^2 + 2x - 3)$$

(c) Solve the equation:

$$\log_5(x^2 + 2x - 3) = 2 \Rightarrow x^2 + 2x - 3 = 5^2 = 25 \Rightarrow x^2 + 2x - 28 = 0$$

Use the quadratic formula:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{(2)^2 + 4 \cdot 28}}{2} = \frac{-2 \pm \sqrt{4 + 112}}{2} = \frac{-2 \pm \sqrt{116}}{2} \\ &= \frac{-2 \pm 2\sqrt{29}}{2} = -1 \pm \sqrt{29} \end{aligned}$$

Check for extraneous solutions:

$$-1 + \sqrt{29} \approx -1 + 5.385 \approx 4.385 \text{ (valid)}, \quad -1 - \sqrt{29} < 0 \text{ (invalid)}$$

Only the positive root satisfies $x > 1$, so:

$x = -1 + \sqrt{29}$

Question 6

The intensity I of an earthquake is measured on the Richter scale using the formula:

$$R = \log_{10} \left(\frac{I}{I_0} \right)$$

where R is the Richter magnitude and $I_0 = 10^{-4}$ is the reference intensity.

- (a) If an earthquake has intensity $I = 10^{-1.5}$, what is its magnitude R ?
- (b) If an earthquake has a Richter magnitude of $R = 6.2$, what is its intensity I ?
- (c) How many times more intense is an earthquake of magnitude 7.2 compared to one of magnitude 5.2?

Question 6 - Solution

(a) Compute the magnitude R :

$$R = \log_{10} \left(\frac{10^{-1.5}}{10^{-4}} \right) = \log_{10}(10^{2.5}) = \boxed{2.5}$$

(b) Find the intensity for $R = 6.2$:

$$R = \log_{10} \left(\frac{I}{10^{-4}} \right) \Rightarrow 10^R = \frac{I}{10^{-4}} \Rightarrow I = 10^R \cdot 10^{-4}$$

$$I = 10^{6.2-4} = 10^{2.2} \approx \boxed{158.49}$$

(c) Compare intensities of magnitudes **7.2** and **5.2**:

Let I_1 and I_2 be the intensities corresponding to magnitudes 7.2 and 5.2, respectively.

$$\log_{10} \left(\frac{I_1}{I_0} \right) = 7.2 \Rightarrow I_1 = 10^{7.2} \cdot I_0, \quad \log_{10} \left(\frac{I_2}{I_0} \right) = 5.2 \Rightarrow I_2 = 10^{5.2} \cdot I_0$$

$$\frac{I_1}{I_2} = \frac{10^{7.2} \cdot I_0}{10^{5.2} \cdot I_0} = 10^{7.2-5.2} = 10^2 = \boxed{100}$$

So the magnitude 7.2 earthquake is **100 times more intense** than the 5.2 earthquake.

Question 7

Solve the following logarithmic equation:

$$2\log_4(x+1) - \log_4(x-3) = 1$$

- (a) Solve the equation algebraically.
- (b) State any domain restrictions and eliminate extraneous solutions.

Question 7 - Solution**(a) Combine the logarithms:**

Use the power rule:

$$\log_4(x+1)^2 - \log_4(x-3) = \log_4\left(\frac{(x+1)^2}{x-3}\right)$$

So the equation becomes:

$$\log_4\left(\frac{(x+1)^2}{x-3}\right) = 1 \Rightarrow \frac{(x+1)^2}{x-3} = 4^1 = 4$$

Multiply both sides by $x-3$:

$$(x+1)^2 = 4(x-3) \Rightarrow x^2 + 2x + 1 = 4x - 12 \Rightarrow x^2 - 2x + 13 = 0$$

Solve with the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(13)}}{2} = \frac{2 \pm \sqrt{4 - 52}}{2} = \frac{2 \pm \sqrt{-48}}{2} = \frac{2 \pm 4i\sqrt{3}}{2} = 1 \pm 2i\sqrt{3}$$

(b) Domain restrictions:

$$x+1 > 0 \Rightarrow x > -1, \quad x-3 > 0 \Rightarrow x > 3$$

So domain is: $x > 3$

The resulting quadratic has no real roots, so the original real logarithmic equation has no solution.

No real solution.

Question 8

Let $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$. Suppose that

$$f(2) = 1 \quad \text{and} \quad f(5) \approx 2.32193.$$

- (a) Estimate $f(10)$ using logarithm properties.
- (b) Estimate $f(5/2)$ using logarithm properties.
- (c) Explain the properties used.

Question 8 - Solution

The product and quotient laws hold for positive arguments:

$$f(10) = f(2) + f(5) \approx 1 + 2.32193 = \boxed{3.32193},$$

$$f(5/2) = f(5) - f(2) \approx 2.32193 - 1 = \boxed{1.32193}.$$

The datum $\log_b 2 = 1$ fixes $b = 2$, so the supplied approximation to $\log_b 5$ is consistent with the same base.

Question 9

A bacteria culture grows according to the function

$$N(t) = N_0 e^{kt}$$

where N_0 is the initial population, k is a constant, and t is the time in hours.

Suppose a culture starts with 500 bacteria and grows to 2000 bacteria in 6 hours.

- (a) Find the growth rate k .
- (b) Use your result to estimate the population after 10 hours.
- (c) After how many hours will the population reach 10,000?

Question 9 - Solution

Growth rate. $2000 = 500e^{6k}$ gives

$$k = \frac{\ln 4}{6} \approx 0.231049 \text{ hr}^{-1}.$$

Population after ten hours.

$$N(10) = 500 \cdot 4^{10/6} \approx 5039.68,$$

so the estimated whole-number population is $\boxed{5040}$.

Time to reach 10000.

$$500e^{kt} = 10000 \quad \implies \quad t = \frac{6 \ln 20}{\ln 4} \approx 12.97 \text{ hours}.$$

Substitution into the exact model verifies both target populations.

Question 10

Solve the logarithmic equation:

$$\ln(x + 2) + \ln(x - 1) = \ln(3x)$$

- (a) Solve the equation algebraically.
- (b) State the domain and eliminate any extraneous solutions.

Question 10 - Solution**(a) Combine the logarithmic expressions:**

Use the product rule:

$$\ln[(x+2)(x-1)] = \ln(3x) \Rightarrow (x+2)(x-1) = 3x \Rightarrow x^2 + x - 2 = 3x \Rightarrow x^2 - 2x - 2 = 0$$

Use the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 + 4(1)(2)}}{2} = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = \boxed{1 \pm \sqrt{3}}$$

(b) Domain restrictions:

We need all logarithms to have positive arguments:

$$x+2 > 0 \Rightarrow x > -2, \quad x-1 > 0 \Rightarrow x > 1, \quad 3x > 0 \Rightarrow x > 0$$

Combining all: $\boxed{x > 1}$ Now test each solution: - $x = 1 + \sqrt{3} \approx 2.73 > 1 \rightarrow$ valid - $x = 1 - \sqrt{3} \approx -0.73 < 1 \rightarrow$ invalid (violates domain)

$$\text{Final answer: } \boxed{x = 1 + \sqrt{3}}$$